

# Large time behaviour of solutions to parabolic equations with Dirichlet operators and nonlinear dependence on measure data

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## Abstract

We study large time behaviour of solutions of the Cauchy problem for equations of the form  $\partial_t u - Lu + \lambda u = f(x, u) + g(x, u) \cdot \mu$ , where  $L$  is the operator associated with a regular lower bounded semi-Dirichlet form  $\mathcal{E}$  and  $\mu$  is a non-negative bounded smooth measure with respect to the capacity determined by  $\mathcal{E}$ . We show that under the monotonicity and some integrability assumptions on  $f, g$  as well as some assumptions on the form  $\mathcal{E}$ ,  $u(t, x) \rightarrow v(x)$  as  $t \rightarrow \infty$  for quasi-every  $x$ , where  $v$  is a solution of some elliptic equation associated with our parabolic equation. We also provide the rate convergence. Some examples illustrating the utility of our general results are given.

**Mathematics Subject Classification (2010):** Primary: 35B40, 35K58; Secondary: 60H30.

**Keywords:** Semilinear equation; Dirichlet operator, Measure data, Large time behaviour of solutions, Rate of convergence, Backward stochastic differential equation.

## 1 Introduction

Let  $E$  be locally compact separable metric space,  $m$  be an everywhere dense Borel measure on  $E$  and let  $L$  be the operator associated with a regular lower bounded semi-Dirichlet form  $(B, V)$  on  $L^2(E; m)$ . The main purpose of the paper is to study large time behaviour of solutions of the Cauchy problem

$$\begin{cases} \partial_t u - Lu + \lambda u = f(x, u) + g(x, u) \cdot \mu & \text{in } (0, \infty) \times E, \\ u(0, \cdot) = \varphi & \text{on } E. \end{cases} \quad (1.1)$$

In (1.1),  $\varphi : E \rightarrow \mathbb{R}$ ,  $f, g : E \times \mathbb{R} \rightarrow \mathbb{R}$  are measurable functions,  $\mu$  is a smooth measure with respect to the capacity associated with  $(B, V)$  and  $\lambda \geq 0$ .

The class of operators corresponding to regular lower bounded Dirichlet forms is quite large. It contains both local operators whose model example is the Laplace

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operator  $\Delta$  or Laplace operator perturbed by the first order operator, as well as nonlocal operators whose model example is the  $\alpha$ -Laplace operator  $(\Delta)^{\alpha/2}$  with  $\alpha \in (0, 2)$  or  $\alpha$ -Laplace operator with variable exponent  $\alpha$  satisfying some regularity conditions. Many interesting examples of operators associated with regular semi-Dirichlet forms are to be found in [10, 14, 17, 21, 24]. In fact our methods also allow to treat equations with operators associated with quasi-regular forms (see remarks at the end of Section 5).

As for the data  $\varphi, f, g$ , we assume that  $\varphi \in L^1(E; m)$ ,  $f, g$  are continuous and monotone in the second variable  $u$  and satisfy mild integrability conditions. Our basic assumption on  $\mu$  is that it is a smooth measure (with respect to the capacity associated with  $(B, V)$ ) of class  $\mathcal{R}^+(E)$ , i.e. a positive smooth measure such that  $E_x A_\infty^\mu < \infty$  for quasi-every (q.e. for short)  $x \in E$ , where  $A^\mu$  is the additive functional of the Hunt process associated with  $(B, V)$  in the Revuz correspondence with  $\mu$ . Equivalently, our condition imposed on  $\mu$  means that the potential (associated with  $(B, V)$ ) of  $\mu$  is  $m$ -a.e. finite. It is known that if the form  $(B, V)$  is transitive or  $\lambda > 0$  then  $\mathcal{R}^+(E)$  contains the class  $\mathcal{M}_{0,b}^+(E)$  of positive bounded smooth measures on  $E$ . In general, the inclusion  $\mathcal{M}_{0,b}^+(E) \subset \mathcal{R}^+(E)$  is strict. Elliptic equations with unbounded measures of class  $\mathcal{R}^+(E)$  are considered for instance in the monograph [22]; see also Section 6.

Let  $v$  be a solution of the elliptic equation

$$-Lv + \lambda v = f(x, v) + g(x, v) \cdot \mu \quad \text{in } E. \quad (1.2)$$

Our main result says that under the assumptions on  $\varphi, f, g$  mentioned before and some additional mild assumptions on the semigroup  $(P_t)$  and the resolvent  $(R_\alpha)$  associated with  $(B, V)$ ,

$$\lim_{t \rightarrow \infty} u(t, x) = v(x) \quad (1.3)$$

for q.e.  $x \in E$ . We also estimate the rate of convergence. Our main estimate says that for every  $q \in (0, 1)$  there is  $C(q) > 0$  such that for q.e.  $x \in E$ ,

$$|u(t, x) - v(x)| \leq 3P_t|\varphi|(x) + 3P_t(R_0(|f(\cdot, 0)| + |g(\cdot, 0)| \cdot \tilde{\mu}))(x), \quad t > 0. \quad (1.4)$$

The quantities on the right hand-side of (1.4) can be estimated for concrete operators  $L$ . We give some examples in Section 6.

To our knowledge, in case  $L$  is a nonlocal operator, our results (1.3), (1.4) are entirely new. In case  $L$  is local, they generalize the results obtained in the paper [11] in which  $g \equiv 1$  and  $L$  is a uniformly elliptic divergence form operator. Note, however, that in [11] systems of equations are treated. We also strengthen slightly the results of [19] concerning asymptotic behaviour of solutions of equations involving Laplace operator  $\Delta$  and absorbing term of the form  $h(u)|\nabla u|^2$  with  $h$  satisfying the “sign condition”. Some other results on asymptotic behaviour, which are not covered by our approach, are to be found in [26, 27, 28]. In [27, 28] equations involving Leray-Lions type operators and smooth measure data are considered while [26] deals with linear equations with general, possibly singular, bounded measure  $\mu$ . Note that the methods used in [26, 27, 28] do not provide estimates between the parabolic solution and the corresponding stationary solution.

In order to prove (1.3), (1.4) we develop the probabilistic approach initiated in [11]. We find interesting that it provides a unified way of treating a wide variety of seemingly disparate examples (see Section 6).

Although in the paper we deal mainly with the asymptotic behaviour for solutions of (1.1), the first question we treat is the existence and uniqueness of solutions of problems (1.1) and (1.2). Here our results are also new, but they proofs rely on our earlier results proved in [14, 17] in case  $g \equiv 1$ . In fact, in the parabolic case we prove the existence and uniqueness of solutions to problems involving operators  $L_t$  and data  $f, g, \mu$  depending on time, i.e. more general then problem (1.1). Finally, let us note that in the paper we consider probabilistic solutions of (1.2) and (1.3) (see Section 3 for the definitions). It is worth pointing out, however, that in the case where  $(B^{(t)}, V)$  are (non-symmetric) Dirichlet forms, the probabilistic solutions coincide with the renormalized solutions defined in [16] (in the elliptic case under the additional assumption that  $(B, V)$  satisfies the strong sector condition and either  $(B, V)$  is transient or  $\lambda > 0$ ). For local operators these renormalized solutions coincide with the usual renormalized solutions (see [8, 29] and also [15]).

## 2 Preliminaries

In the paper  $E$  is a locally compact separable metric space,  $E^1 = \mathbb{R} \times E$ ,  $m$  is an everywhere dense Borel measure on  $E$  and  $m_1 = dt \otimes m$ . For  $T > 0$  we write  $E_T = [0, T] \times E$ ,  $E_{0,T} = (0, T] \times E$ . By  $\mathcal{B}_b(E)$  we denote the set of all real bounded Borel measurable functions on  $E$  and by  $\mathcal{B}_b^+(E)$  we denote the subset of  $\mathcal{B}_b(E)$  consisting of all nonnegative functions. The sets  $\mathcal{B}_b(E^1)$ ,  $\mathcal{B}_b^+(E^1)$  are defined analogously.

### 2.1 Dirichlet forms

Let  $H = L^2(E; m)$  and let  $(\cdot, \cdot)$  denote the usual inner product in  $H$ . We assume that we are given a family  $\{B^{(t)}, t \in [0, T]\}$  of regular semi-Dirichlet forms on  $H$  with common domain  $V \subset H$  (see [24, Section 1.1]). We assume that the forms  $B^{(t)}$  are lower bounded and satisfy the sector condition with constants  $\alpha_0 \geq 0$ ,  $K \geq 1$  independent of  $t \in [0, T]$ . Let us recall that this means that

$$B_{\alpha_0}^{(t)}(\varphi, \psi) \geq 0, \quad \varphi, \psi \in V,$$

where  $B_{\lambda}^{(t)}(\varphi, \psi) = B^{(t)}(\varphi, \psi) + \lambda(\varphi, \psi)$  for  $\lambda \geq 0$ , and that

$$|B_{\alpha_0}^{(t)}(\varphi, \psi)| \leq K B_{\alpha_0}^{(t)}(\varphi, \varphi)^{1/2} B_{\alpha_0}^{(t)}(\psi, \psi)^{1/2}, \quad \varphi, \psi \in V$$

for all  $t \in [0, T]$ . We also assume that  $[0, T] \ni t \mapsto B^{(t)}(\varphi, \psi)$  is Borel measurable for every  $\varphi, \psi \in V$  and there is  $c \geq 1$  such that

$$c^{-1} B_{\alpha_0}(\varphi, \varphi) \leq B_{\alpha_0}^{(t)}(\varphi, \varphi) \leq c B_{\alpha_0}(\varphi, \varphi), \quad t \in [0, T], \varphi \in V, \quad (2.1)$$

where  $B(\varphi, \varphi) = B^{(0)}(\varphi, \varphi)$ . By putting  $B^{(t)} = B$  for  $t \notin [0, T]$  we may and will assume that  $B^{(t)}$  is defined and satisfies (2.1) for all  $t \in \mathbb{R}$ . As usual, by  $\tilde{B}^{(t)}$  we denote the symmetric part of  $B^{(t)}$ , i.e.  $\tilde{B}^{(t)}(\varphi, \psi) = \frac{1}{2}(B^{(t)}(\varphi, \psi) + B^{(t)}(\psi, \varphi))$ .

Note that by the assumption  $V$  is a dense subspace of  $H$  and the form  $(B, V)$  is closed, i.e.  $V$  is a real Hilbert space with respect to  $\tilde{B}_1(\cdot, \cdot)$ , which is densely and continuously embedded in  $H$ . By  $\|\cdot\|_V$  we denote the norm in  $V$ , i.e.  $\|\varphi\|_V^2 = B_1(\varphi, \varphi)$ ,

$\varphi \in V$ . By  $V'$  we denote the dual space of  $V$  and by  $\|\cdot\|_{V'}$  the corresponding norm. We set  $\mathcal{H} = L^2(\mathbb{R}; H)$ ,  $\mathcal{V} = L^2(\mathbb{R}; V)$ ,  $\mathcal{V}' = L^2(\mathbb{R}; V')$  and

$$\|u\|_{\mathcal{V}}^2 = \int_{\mathbb{R}} \|u(t)\|_V^2 dt, \quad \|u\|_{\mathcal{V}'}^2 = \int_{\mathbb{R}} \|u(t)\|_{V'}^2 dt. \quad (2.2)$$

We shall identify  $H$  and its dual  $H'$ . Then  $V \subset H \simeq H' \subset V'$  continuously and densely, and hence  $\mathcal{V} \subset \mathcal{H} \simeq \mathcal{H}' \subset \mathcal{V}'$  continuously and densely.

For  $u \in \mathcal{V}$  we denote by  $\frac{\partial u}{\partial t}$  the derivative in the distribution sense of the function  $t \mapsto u(t) \in V$  and we set

$$\mathcal{W} = \{u \in \mathcal{V}; \frac{\partial u}{\partial t} \in \mathcal{V}'\}, \quad \|u\|_{\mathcal{W}} = \|u\|_{\mathcal{V}} + \|\frac{\partial u}{\partial t}\|_{\mathcal{V}'} \quad (2.3)$$

By  $\mathcal{E}$  we denote the time dependent Dirichlet form associated with the family  $\{(B^{(t)}, V), t \in \mathbb{R}\}$ , that is

$$\mathcal{E}(u, v) = \begin{cases} \langle -\frac{\partial u}{\partial t}, v \rangle + \mathcal{B}(u, v), & u \in \mathcal{W}, v \in \mathcal{V}, \\ \langle \frac{\partial v}{\partial t}, u \rangle + \mathcal{B}(u, v), & u \in \mathcal{V}, v \in \mathcal{W}, \end{cases} \quad (2.4)$$

where  $\langle \cdot, \cdot \rangle$  is the duality pairing between  $\mathcal{V}'$  and  $\mathcal{V}$  and

$$\mathcal{B}(u, v) = \int_{\mathbb{R}} B^{(t)}(u(t), v(t)) dt. \quad (2.5)$$

Note that  $\mathcal{E}$  can be identified with some generalized Dirichlet form (see [33, Example I.4.9(iii)]).

Given a time dependent form (2.4) we define quasi notions with respect to  $\mathcal{E}$  (exceptional sets, nests, quasi-continuity as in [24, Section 6.2]. Note that by [24, Theorem 6.2.11] each element  $u$  of  $\mathcal{W}$  has a quasi-continuous  $m_1$ -version. We will denote it by  $\tilde{u}$ . Quasi-notions with respect to  $(B, V)$  are defined as in [24, Section 2.2].

By  $S(E)$  we denote the set of all smooth measures on  $E$  with respect to the form  $(B, V)$  (see, e.g., [24, Section 4.1] for the definition). By  $S(E^1)$  we denote the set of all smooth measures on  $E^1$  with respect to  $\mathcal{E}$  (see [13]) and by  $S(E_{0,T})$  the set of all smooth measures on  $E^1$  with support in  $E_{0,T}$ . By  $\mathcal{M}_b(E_{0,T})$  we denote the set of all signed Borel measures on  $E^1$  with support in  $E_{0,T}$  such that  $|\mu|(E^1) < \infty$ , where  $|\mu|$  stand for the total variation of  $\mu$ .  $\mathcal{M}_{0,b}(E_{0,T})$  is the subset of  $\mathcal{M}_b(E_{0,T})$  consisting of all smooth measures. Analogously we define the classes  $\mathcal{M}_b(E)$ ,  $\mathcal{M}_{0,b}(E)$ .

We will say that a Borel measure  $\mu$  on  $E^1$  does not depend on time if it is of the form

$$\mu = dt \otimes \tilde{\mu} \quad (2.6)$$

for some Borel measure  $\tilde{\mu}$  on  $E$ . Since  $\tilde{\mu}(B) = \mu([0, 1] \times B)$  for  $B \in \mathcal{B}(E)$ ,  $\tilde{\mu}$  is uniquely determined by  $\mu$ . From now on given  $\mu$  not depending on time we will denote by  $\tilde{\mu}$  the Borel measure on  $E$  determined by (2.6).

**Lemma 2.1.** *If  $\mu \in S(E_{0,T})$  does not depend on time then  $\tilde{\mu} \in S(E)$ .*

*Proof.* Let  $\alpha > \alpha_0$  and let  $\text{Cap}$  denote the capacity associated with the form  $B_\alpha$  defined in [24, Section 2.1], whereas  $\text{CAP}$  denote the capacity associated with  $\mathcal{E}$  defined in [24, Section 6.2]. It is enough to prove that for every  $A \subset E$ , if  $\text{Cap}(A) = 0$  then

$\text{CAP}([0, T] \times A) = 0$ . Suppose that  $\text{Cap}(A) = 0$ . Then by [24, Eq. (2.1.8)], for every  $\varepsilon > 0$  there exists an open set  $U_\varepsilon \subset E$  and  $\psi_\varepsilon \in V$  such that  $A \subset U_\varepsilon$ ,  $\psi_\varepsilon \geq 1$  on  $U_\varepsilon$  and

$$B_\alpha(\psi_\varepsilon, \psi_\varepsilon) \leq \text{Cap}(U_\varepsilon) \leq \varepsilon.$$

By the above inequality and (2.1),

$$B_\alpha^{(t)}(\psi_\varepsilon, \psi_\varepsilon) \leq c\varepsilon, \quad t \in \mathbb{R}. \quad (2.7)$$

Let  $f$  be a continuous function on  $\mathbb{R}$  with compact support such that  $f \geq 1$  on  $[-T, 2T]$  and let  $\eta_\varepsilon = f\psi_\varepsilon$ . Then  $\eta_\varepsilon \in \mathcal{W}$  and by [24, Eq. (6.2.21)] and (2.7),

$$\text{CAP}([0, T] \times A) \leq C(\|\frac{\partial \eta_\varepsilon}{\partial t}\|_{L^2(0, T; H)}^2 + \mathcal{B}_\alpha(\eta_\varepsilon, \eta_\varepsilon)) \leq \varepsilon C' T(\|\frac{\partial f}{\partial t}\|_\infty^2 + \|f\|_\infty^2),$$

where  $C' > 0$  depends only on  $c$  and  $\alpha$ . Since  $\varepsilon > 0$  was arbitrary, the desired result follows.  $\square$

## 2.2 Markov processes and additive functionals

In what follows  $\partial$  is a one-point compactification of  $E$ . If  $E$  is already compact then we adjoin  $\partial$  to  $E$  as an isolated point. When considering Dirichlet forms, we adopt the convention that every function  $f$  on  $E$  is extended to  $E \cup \{\partial\}$  by setting  $f(\partial) = 0$ . When considering time dependent Dirichlet forms, we adopt the convention that every function  $\varphi$  on  $E$  is extended to  $E^1$  by setting  $\varphi(t, x) = \varphi(x)$ ,  $(t, x) \in E^1$ , and every function  $f$  on  $E^1$  (resp.  $E_{0, T}$ ) is extended to  $E^1 \cup \{\partial\}$  by setting  $f(\partial) = 0$  (resp.  $f(z) = 0$  for  $z \in E^1 \cup \{\partial\} \setminus E_{0, T}$ ).

Let  $\mathcal{E}$  be the form defined by (2.4). By [24, Theorem 6.3.1] there exists a Hunt process  $\mathbf{M} = (\Omega, (\mathcal{F}_t)_{t \geq 0}, (\mathbf{X}_t)_{t \geq 0}, (P_z)_{z \in E^1 \cup \{\partial\}})$  with state space  $E^1$ , life time  $\zeta$  and cemetery state  $\partial$  associated with  $\mathcal{E}$  in the resolvent sense, i.e. for every  $\alpha > 0$  and  $f \in L^2(E^1; m_1) \cap \mathcal{B}_b(E^1)$  the resolvent of  $\mathbf{M}$  defined as

$$\mathbf{R}_\alpha f(z) = \int_0^\infty e^{-\alpha t} E_z f(\mathbf{X}_t) dt, \quad z \in E^1, f \in \mathcal{B}_b(E^1)$$

is an  $\mathcal{E}$ -quasi-continuous  $m_1$ -version of the resolvent associated with the form  $\mathcal{E}$ . By [24, Theorem 6.3.1], if

$$\mathbf{X}_t = (\tau(t), X_{\tau(t)}), \quad t \geq 0 \quad (2.8)$$

is a decomposition of  $\mathbf{X}$  into the process on  $\mathbb{R}$  and on  $E$  then  $\tau$  is the uniform motion to the right, i.e.  $\tau(t) = \tau(0) + t$ ,  $\tau(0) = s$ ,  $P_z$ -a.s. for  $z = (s, x) \in E^1$ . Moreover, one can check that for every  $s \in \mathbb{R}$  the process  $\mathbb{M}^{(s)} = (\Omega, (\mathcal{F}_{s+t})_{t \geq 0}, (X_{s+t})_{t \geq 0}, (P_{s,x})_{x \in E \cup \{\partial\}})$  is a Hunt process with life time  $\xi^s = \inf\{t \geq 0 : X_{s+t} \in \partial\}$  associated with the form  $(B^{(s)}, V)$ .

Let us recall that an additive functional (AF for short) of  $\mathbf{M}$  is called natural if  $A$  and  $\mathbf{M}$  have no common discontinuities. It is known (see [13, Section 2]) that for every  $\mu \in S(E^1)$  there exists a unique positive natural AF  $A$  of  $\mathbf{M}$  such that  $A$  is in the Revuz correspondence with  $\mu$ , i.e. for every  $m_1$ -integrable  $\alpha$ -coexcessive function  $h$  with  $\alpha > 0$ ,

$$\lim_{\beta \rightarrow \infty} \beta E_{h \cdot m_1} \int_0^\infty e^{-(\alpha+\beta)t} f(\mathbf{X}_t) dA_t = \int_{E^1} f(z) h(z) \mu(dz), \quad f \in \mathcal{B}_b^+(E^1),$$

where  $E_{h \cdot m_1}$  denotes the expectation with respect to  $P_{h \cdot m_1}(\cdot) = \int_{E^1} P_z(\cdot) h(z) m_1(dz)$ . In what follows we will denote it by  $A^\mu$ . On the contrary, if  $A$  is a positive natural AF of  $\mathbf{M}$  then modifying the proof of [10, Lemma 5.1.7] (we replace quasi-notions and facts used in the proof in [10] by corresponding quasi-notions and facts from [24, Sections 2–4]; for the case of (non-symmetric) Dirichlet form see also [23, Theorem 5.6]) one can show that there exists a smooth measure on  $E^1$  such that  $A$  is in the Revuz correspondence with  $\mu$ .

We set

$$\mathcal{R}(E_{0,T}) = \{\mu : |\mu| \in S(E_{0,T}), E_z \int_0^{\zeta_\tau} dA_t^{|\mu|} < \infty \text{ for } m_1\text{-a.e. } z \in E_{0,T}\},$$

where

$$\zeta_\tau = \zeta \wedge (T - \tau(0)).$$

By [13, Proposition 3.4], in the definition of  $\mathcal{R}(E_{0,T})$  one can replace  $m_1$ -a.e. by q.e. (with respect to  $\mathcal{E}$ ). By [13, Proposition 3.8], if  $(B, V)$  is a (non-symmetric) Dirichlet form or, more generally, a semi-Dirichlet form satisfying the duality condition (see [13] for the definition), then  $\mathcal{M}_{0,b}(E_{0,T}) \subset \mathcal{R}(E_{0,T})$ . The inclusion may be strict (see [13, Example 5.2]).

Let  $\mu \in S(E)$ . Since  $\mathbb{M}^{(0)}$  corresponds to  $(B, V)$ , by [24, Theorem 4.1.16] there is a unique positive continuous AF  $A^{0,\mu}$  of  $\mathbb{M}^{(0)}$  such that  $A^{0,\mu}$  is in the Revuz correspondence with  $\mu$ , i.e.

$$\lim_{\alpha \rightarrow \infty} \alpha E_m \int_0^\infty e^{-\alpha t} f(X_t) dA_t^{0,\mu} = \int_E f(x) \mu(dx), \quad f \in \mathcal{B}_b^+(E).$$

We set

$$\mathcal{R}(E) = \{\mu : |\mu| \in S(E), E_{0,x} \int_0^\zeta dA_t^{0,|\mu|} < \infty \text{ for } m\text{-a.e. } x \in E\}$$

By [14, Lemma 4.2] in the above definition of the class  $\mathcal{R}(E)$  one can replace  $m$ -a.e. by q.e. (with respect to  $(B, V)$ ) and that by [17, Proposition 3.2], if  $(B, V)$  is a transient (non-symmetric) Dirichlet form then  $\mathcal{M}_{0,b}(E) \subset \mathcal{R}(E)$ . In general the inclusion is strict (see remarks following [17, Proposition 3.2]).

While considering elliptic equations and large time behaviour of parabolic equations we will assume that

$$B^{(t)}(\varphi, \psi) = B(\varphi, \psi), \quad \varphi, \psi \in V, \quad t \in \mathbb{R}. \quad (2.9)$$

**Lemma 2.2.** *Assume (2.9).*

- (i) *For every  $s \geq 0$  the distribution of  $(X \circ \theta_{\tau(0)}, A^{0,\tilde{\mu}} \circ \theta_{\tau(0)})$  under  $P_{s,x}$  is equal to the distribution of  $(X, A^{0,\tilde{\mu}})$  under  $P_{0,x}$ .*
- (ii)  *$A^\mu = A^{0,\tilde{\mu}} \circ \theta_{\tau(0)}$ .*

*Proof.* (i) First suppose that  $\tilde{\mu}(dx) = f(x) m(dx)$  for some  $f \in L^1(E; m)$ . Then  $A_t^{0,\tilde{\mu}} = \int_0^t f(X_r) dr$  and hence  $A_t^{0,\tilde{\mu}} \circ \theta_{\tau(0)} = \int_0^t f(X_r \circ \theta_{\tau(0)}) dr$ . Therefore (i) follows from the fact that the distribution of  $X$  under  $P_{0,x}$  is equal to the distribution of  $X \circ \theta_{\tau(0)}$  under  $P_{s,x}$ . Now assume that  $\mu$  belongs to the set  $S_0(E)$  of smooth measures of finite

energy. Then  $A_t^{0,\tilde{\mu}} = \int_0^t e^r \tilde{A}_r$ , where  $\tilde{A}_t = \lim_{n \rightarrow \infty} \tilde{A}_t^n$  and  $\tilde{A}_t^n = \int_0^t e^{-r} f_n(X_r) dr$  for some  $f_n \in L^1(E; m)$  (see the proof of [10, Theorem 5.1.1] or [24, Theorem 4.1.10]). From this and the first part we deduce that (i) is satisfied for every  $\tilde{\mu} \in S_0(E)$ . By [24, Lemma 4.1.14] there exists a nest  $\{F_n\}$  such that  $\mathbf{1}_{F_n} \cdot \tilde{\mu} \in S_0(E)$  for each  $n \in \mathbb{N}$ . Since we already know that (i) holds for  $\tilde{\mu}$  replaced by  $\mathbf{1}_{F_n} \cdot \tilde{\mu}$ , applying the monotone convergence theorem we conclude that it holds for  $\tilde{\mu}$  replaced by  $\mathbf{1}_{\bigcup_{n=1}^\infty F_n} \cdot \tilde{\mu}$ , and hence for  $\tilde{\mu}$ , because the set  $E \setminus \bigcup_{n=1}^\infty F_n$  is exceptional.

(ii) Let  $A = A^{0,\tilde{\mu}} \circ \theta_{\tau(0)}$ . Under (2.9) the distribution of  $A$  under  $P_{s,x}$  is equal to the distribution of  $A^{0,\tilde{\mu}}$  under  $P_{0,x}$ . Hence

$$E_{s,x} \int_0^\infty e^{-\alpha t} dA_t = E_{s,x} \int_0^\infty e^{-\alpha t} d(A_t^{0,\tilde{\mu}} \circ \theta_s) = E_{0,x} \int_0^\infty e^{-\alpha t} dA_t^{0,\tilde{\mu}} =: R_\alpha \tilde{\mu}(x).$$

One can check that  $A$  is a CAF of  $\mathbf{M}$ . Let  $\nu$  denote its Revuz measure. Then for every  $f$  of the form  $f = \xi g$  with  $\xi \in \mathcal{B}_b^+(\mathbb{R})$ ,  $g \in \mathcal{B}_b^+(E)$  we have

$$\begin{aligned} \int_{E^1} f(z) \nu(dz) &= \lim_{\alpha \rightarrow \infty} \alpha \int_{E^1} \left( f(z) E_z \int_0^\infty e^{-\alpha t} dA_t \right) m_1(dz) \\ &= \lim_{\alpha \rightarrow \infty} \alpha \int_{E^1} \xi(s) g(x) R_\alpha \tilde{\mu}(x) ds m(dx) \\ &= \int_{\mathbb{R}} \xi(s) ds \cdot \int_E g(x) \tilde{\mu}(dx) = \int_{E^1} f(s, x) ds \tilde{\mu}(dx). \end{aligned}$$

Hence  $\nu = dt \otimes \tilde{\mu} = \mu$ . Since additive functionals are uniquely determined by their Revuz measures, this proves (ii).  $\square$

### 3 Parabolic PDEs and generalized BSDEs

For  $t \in [0, T]$  let  $L_t$  denote the operator associated with the form  $(B^{(t)}, V)$ , i.e.

$$D(L_t) = \{u \in V : v \mapsto \mathcal{E}(u, v) \text{ is continuous with respect to } (\cdot, \cdot)_H^{1/2} \text{ on } V\}$$

and

$$(-L_t \varphi, \psi) = B^{(t)}(\varphi, \psi), \quad \varphi \in D(L_t), \psi \in V \quad (3.1)$$

(see [21, Proposition I.2.16]). Suppose we are given measurable functions  $\varphi : E \rightarrow \mathbb{R}$ ,  $f, g : E_T \times \mathbb{R} \rightarrow \mathbb{R}$  and  $\mu \in \mathcal{R}(E_{0,T})$ . In this section we consider the following Cauchy problems with terminal and initial conditions:

$$\partial_t u + L_t u = -f(t, x, u) - g(t, x, u) \cdot \mu, \quad u(T) = \varphi \quad (3.2)$$

and

$$\partial_t u - L_t u = f(t, x, u) + g(t, x, u) \cdot \mu, \quad u(0) = \varphi. \quad (3.3)$$

**Definition.** Let  $z \in E_T$ . We say that a pair  $(Y^z, M^z)$  is a solution of the BSDE

$$Y_t^z = \varphi(\mathbf{X}_{\zeta_\tau}) + \int_{t \wedge \zeta_\tau}^{\zeta_\tau} f(\mathbf{X}_r, Y_r^z) dr + \int_{t \wedge \zeta_\tau}^{\zeta_\tau} g(\mathbf{X}_r, Y_r^z) dA_r^\mu - \int_{t \wedge \zeta_\tau}^{\zeta_\tau} dM_r^z, \quad t \geq 0 \quad (3.4)$$

on the space  $(\Omega, \mathcal{F}, P_z)$  if

- (a)  $Y^z$  is an  $(\mathcal{F}_t)$ -progressively measurable process of class D under  $P_z$ ,  $M^z$  is an  $(\mathcal{F}_t)$ -martingale under  $P_z$  such that  $M_0^z = 0$ ,
- (b)  $\int_0^{\zeta_\tau} |f(\mathbf{X}_t, Y_t^z)| dt < \infty$ ,  $\int_0^{\zeta_\tau} |g(\mathbf{X}_t, Y_t^z)| d|A^\mu|_t < \infty$ ,  $P_z$ -a.s. (Here  $|A^\mu|_t$  denotes the variation of the process  $A^\mu$  on  $[0, t]$ ),
- (c) Eq. (3.4) is satisfied  $P_z$ -a.s.

Let us recall that a càdlàg  $(\mathcal{F}_t)$ -adapted process  $Y$  is of Doob's class D under  $P_z$  if the collection  $\{Y_\tau : \tau \in \mathcal{T}\}$ , where  $\mathcal{T}$  is the set of all finite valued  $(\mathcal{F}_t)$ -stopping times, is uniformly integrable under  $P_z$ . Let  $\mathcal{L}^1(P_z)$  denote the space of càdlàg  $(\mathcal{F}_t)$ -adapted processes  $Y$  with finite norm

$$\|Y\|_{z,1} = \sup\{E_z|Y_\tau| : \tau \in \mathcal{T}\}.$$

It is known that  $\mathcal{L}^1(P_z)$  is complete (see [9, p. 90]). Moreover, if processes  $Y^n$  are of class D and  $Y^n \rightarrow Y$  in  $\mathcal{L}^1(P_z)$  then  $Y$  is of class D. To see this, let us fix  $\varepsilon > 0$  and choose  $n$  so that  $\|Y^n - Y\|_{z,1} \leq \varepsilon/2$ . Since the family  $\{Y_\tau^n\}$  is of class D, there exists  $\delta > 0$  such that if  $P_z(A) < \delta$  then  $\int_A |Y_\tau^n| dP_z < \varepsilon/2$ . It follows that if  $P_z(A) < \delta$  then for every finite  $(\mathcal{F}_t)$ -stopping time  $\tau$ ,

$$\int_A |Y_\tau| dP_z \leq E_z|Y_\tau^n - Y_\tau| + \int_A |Y_\tau^n| dP_z \leq \varepsilon,$$

which shows that  $\{Y_\tau\}$  is uniformly integrable (see [32, Theorem I.11]).

To simplify notation, in what follows we write

$$f_u(t, x) = f(t, x, u(t, x)), \quad g_u(t, x) = g(t, x, u(t, x)).$$

**Definition.** (a) We say that  $u : E_{0,T} \rightarrow \mathbb{R}$  is a solution of problem (3.2) if  $f_u \cdot m \in \mathcal{R}(E_{0,T})$ ,  $g_u \cdot \mu \in \mathcal{R}(E_{0,T})$  and for q.e.  $z \in E_{0,T}$ ,

$$u(z) = E_z \left( \varphi(\mathbf{X}_{\zeta_\tau}) + \int_0^{\zeta_\tau} f_u(\mathbf{X}_t) dt + \int_0^{\zeta_\tau} g_u(\mathbf{X}_t) dA_t^\mu \right). \quad (3.5)$$

(b) We say that  $u : [0, T] \times E \rightarrow \mathbb{R}$  is a solution of problem (3.3) if  $\bar{u}$  defined as

$$\bar{u}(t, x) = u(T - t, x), \quad (t, x) \in E_{0,T}$$

is a solution of the Cauchy problem with terminal condition of the form

$$\partial_t \bar{u} + L_{T-t} \bar{u} = -f(T - t, x, \bar{u}) - g(T - t, x, \bar{u}) \cdot (\mu \circ \iota_T^{-1}), \quad \bar{u}(T) = \varphi, \quad (3.6)$$

where  $\iota_T : E_T \rightarrow E_T$ ,  $\iota_T(t, x) = (T - t, x)$ .

**Remark 3.1.** If equation (3.6) has the uniqueness property (i.e. has a unique solution  $v_T$  for every  $T > 0$ ) then for every  $a > 0$ ,

$$\bar{v}_T(t, x) := v_T(T - t, x) = v_{T+a}(T + a - t, x) =: \bar{v}_{T+a}(t, x), \quad (t, x) \in [0, T] \times E. \quad (3.7)$$

To see this, let us write  $f_{v_T}^T(x, t) = f(T - t, x, v_T(t, x))$ ,  $g_{v_T}^T(x, t) = g(T - t, x, v_T(t, x))$ . With this notation,

$$\frac{\partial v_T}{\partial t} + L_{T-t} v_T = f_{v_T}^T + g_{v_T}^T \cdot (\mu \circ \iota_T^{-1}), \quad v_T(T) = \varphi$$



and

$$\frac{\partial v_{T+a}}{\partial t} + L_{T+a-t} v_{T+a} = f_{v_{T+a}}^{T+a} + g_{v_{T+a}}^{T+a} \cdot (\mu \circ \iota_{T+a}^{-1}), \quad v_{T+a}(T+a) = \varphi. \quad (3.8)$$

Of course, (3.7) will be proved once we show that

$$v_T(t, x) = v_{T+a}(a+t, x), \quad (t, x) \in E_{0,T}. \quad (3.9)$$

It is known (see [13, p. 1213]) that there exists a generalized nest  $\{F_n\}$  on  $E_{0,T+a}$  such that  $\Phi^{n,T+a} := \mathbf{1}_{F_n} \cdot (f_{v_{T+a}}^{T+a} + g_{v_{T+a}}^{T+a} \cdot (\mu \circ \iota_{T+a}^{-1})) \in S_0(E_{0,T+a})$  for each  $n \in \mathbb{N}$ . Let  $v_{T+a}^n$  denote the solution of the linear equation

$$\frac{\partial v_{T+a}^n}{\partial t} + L_{T+a-t} v_{T+a}^n = \Phi^{n,T+a}, \quad v_{T+a}^n(T+a) = \varphi \quad (3.10)$$

and let

$$v_{T+a,a}^n(t, x) = v_{T+a}^n(a+t, x), \quad (t, x) \in (-a, T] \times E. \quad (3.11)$$

By [13, Theorem 3.7],  $v_{T+a}^n$  is a weak solution of (3.10). Therefore making a simple change of variables shows that  $v_{T+a,a}^n$  is a weak solution of the linear equation

$$\frac{\partial v_{T+a,a}^n}{\partial t} + L_{T-t} v_{T+a,a}^n = \mathbf{1}_{F_n}^a \cdot (f_{v_{T+a,a}}^T + g_{v_{T+a,a}}^T \cdot (\mu \circ \iota_T^{-1})), \quad v_{T+a,a}^n(T) = \varphi, \quad (3.12)$$

where  $\mathbf{1}_{F_n}^a(t, x) = \mathbf{1}_{F_n}(t+a, x)$ . Using the probabilistic representation of the solution of (3.10) and the fact that  $\{F_n\}$  is a nest one can easily show that  $v_{T+a}^n \rightarrow v_{T+a}$  pointwise as  $n \rightarrow \infty$ . Similarly, using the probabilistic representation of the solution of (3.12) one can show that  $v_{T+a,a}^n$  converges pointwise as  $n \rightarrow \infty$  to the solution of (3.8), that is to  $v_T$ . This and (3.11) imply (3.9).

In the rest of this section we say that some property is satisfied quasi-everywhere (q.e. for brevity) if the set of those  $z \in E^1$  for which it does not hold is exceptional with respect to the form  $\mathcal{E}$ .

In what follows we will say that a Borel measurable  $F : E_{0,T} \rightarrow \mathbb{R}$  is  $\mu$ -quasi-integrable ( $F \in qL^1(E_{0,T}; \mu)$  in notation) if  $P_z(\int_0^{\zeta_\tau} |F(\mathbf{X}_t)| dA_t^\mu < \infty) = 1$  for q.e.  $z \in E_{0,T}$ .

Let us remark that if  $\mu = m_1$  then  $A_t^\mu = t$ ,  $t \geq 0$ , so  $m_1$ -quasi-integrability coincides with the notion of quasi-integrability considered in [13, Section 5]) (see also [12, Section 2]).

Our basic assumptions on the data are the following.

- (P1)  $\varphi \in L^1(E; m)$ ,  $\mu \in \mathcal{R}^+(E_{0,T})$ .
- (P2)  $f(\cdot, \cdot, y), g(\cdot, \cdot, y)$  are measurable for every  $y \in \mathbb{R}$  and  $f(t, x, \cdot), g(t, x, \cdot)$  are continuous for every  $(t, x) \in E_{0,T}$ .
- (P3) There is  $\alpha \in \mathbb{R}$  such that  $\langle f(t, x, y) - f(t, x, y'), y - y' \rangle \leq \alpha |y - y'|^2$  for all  $y, y' \in \mathbb{R}$  and  $(t, x) \in E_{0,T}$ .
- (P4)  $f(\cdot, \cdot, 0) \cdot m_1 \in \mathcal{R}(E_{0,T})$  and  $(t, x) \mapsto f(t, x, y) \in qL^1(E_{0,T}; m_1)$  for every  $y \in \mathbb{R}$ .
- (P5)  $\langle g(t, x, y) - g(t, x, y'), y - y' \rangle \leq 0$  for all  $y, y' \in \mathbb{R}$  and  $(t, x) \in E_{0,T}$ .

(P6)  $g(\cdot, \cdot, 0) \cdot \mu \in \mathcal{R}(E_{0,T})$  and  $(t, x) \mapsto g(t, x, y) \in qL^1(E_{0,T}; \mu)$  for every  $y \in \mathbb{R}$ .

In what follows by  $\mathcal{D}^q(P_z)$ ,  $q > 0$ , we denote the space of all  $(\mathcal{F}_t)$ -progressively measurable càdlàg processes  $Y$  such that  $E_z \sup_{t \geq 0} |Y_t|^q < \infty$ .

**Theorem 3.2.** *Assume that (P1)–(P6) are satisfied and  $A^\mu$  is continuous.*

(i) *There exists a unique solution  $u$  of problem (3.2).*

(ii) *Let*

$$M_t^z = E_z \left( \varphi(\mathbf{X}_{\zeta_\tau}) + \int_0^{\zeta_\tau} f_u(\mathbf{X}_s) ds + \int_0^{\zeta_\tau} g_u(\mathbf{X}_s) dA_s^\mu | \mathcal{F}_t \right) - u(\mathbf{X}_0).$$

*Then there exists a càdlàg  $(\mathcal{F}_t)$ -adapted process  $M$  such that  $M_t = M_t^z$ ,  $t \in [0, T]$ ,  $P_z$ -a.s. for q.e.  $z \in E_{0,T}$  and for q.e.  $z \in E_{0,T}$  the pair  $(u(\mathbf{X}), M)$  is a unique solution of (3.4) on the space  $(\Omega, \mathcal{F}, P_z)$ . Moreover,  $u(\mathbf{X}) \in \mathcal{D}^q(P_z)$  for  $q \in (0, 1)$  and  $M$  is a uniformly integrable martingale under  $P_z$  for q.e.  $z \in E_{0,T}$ . Finally, for q.e.  $z \in E_{0,T}$ ,*

$$\begin{aligned} E_z \int_0^{\zeta_\tau} f_u(\mathbf{X}_t) dt + \int_0^{\zeta_\tau} g_u(\mathbf{X}_t) dA_t^\mu \\ \leq E_z \left( \varphi(\mathbf{X}_{\zeta_\tau}) + 2 \int_0^{\zeta_\tau} |f(\mathbf{X}_t, 0)| dt + 3 \int_0^{\zeta_\tau} |g(\mathbf{X}_t, 0)| dA_t^\mu \right). \end{aligned} \quad (3.13)$$

*Proof.* By using the standard change of variables (see, e.g., the beginning of the proof of [3, Lemma 3.1]), without loss of generality we may and will assume that  $\alpha \leq 0$  in condition (P3).

We first prove (ii). The uniqueness of a solution of BSDE (3.4) follows from (P3), (P5) and the fact that  $\mu$  is nonnegative. The proof is standard. We may argue for instance as in the proof of [14, Proposition 2.1] with obvious changes. The proof of the existence of a solution we divide into 2 steps.

Step 1. Let  $\xi = \varphi(\mathbf{X}_{\zeta_\tau})$ ,  $f(t, y) = f(\mathbf{X}_t, y)$ ,  $g(t, y) = g(\mathbf{X}_t, y)$  and let  $A$  be a continuous increasing  $(\mathcal{F}_t)$ -adapted process. Assume that

$$T \cdot \sup_{0 \leq t \leq T} |f(t, 0)| + A_T \cdot \sup_{0 \leq t \leq T} |g(t, 0)| + |\xi| \leq c$$

for some  $c > 0$  and write  $\bar{f}_c(t, y) = f(t, T_c(y))$ ,  $\bar{g}_c(t, y) = g(t, T_c(y))$  where

$$T_c(y) = ((-c) \vee y) \wedge c, \quad y \in \mathbb{R}. \quad (3.14)$$

Then modifying slightly the proof of [14, Lemma 2.6] we show that there exists a unique solution  $(Y, M)$  of the BSDE

$$\begin{aligned} Y_t = \xi + \int_{t \wedge \zeta_\tau}^{\zeta_\tau} (\bar{f}_c(s, Y_s) ds + \int_{t \wedge \zeta_\tau}^{\zeta_\tau} (\bar{g}_c(s, Y_s) - g(s, 0)) dA_s^\mu \\ + \int_{t \wedge \zeta_\tau}^{\zeta_\tau} \bar{g}_c(s, 0) dA_s - \int_{t \wedge \zeta_\tau}^{\zeta_\tau} dM_s, \quad t \geq 0 \end{aligned} \quad (3.15)$$

on the space  $(\Omega, \mathcal{F}, P_z)$  (for brevity, in notation we drop the dependence of  $Y, M$  on  $z$ ). Let  $\text{sgn}(x) = 1$  if  $x > 0$  and  $\text{sgn}(x) = -1$  if  $x \leq 0$ . By the Meyer-Tanaka formula (see [32, p. 216]) and the fact that  $A^\mu$  is continuous,

$$\begin{aligned} |Y_t| &\leq |Y_{\zeta_\tau}| - \int_{t \wedge \zeta_\tau}^{\zeta_\tau} \text{sgn}(Y_s) dY_s \\ &= |\xi| + \int_{t \wedge \zeta_\tau}^{\zeta_\tau} \text{sgn}(Y_s) (\bar{f}_c(s, Y_s) - f(s, 0)) ds + \int_{t \wedge \zeta_\tau}^{\zeta_\tau} \text{sgn}(Y_s) f(s, 0) ds \\ &\quad + \int_{t \wedge \zeta_\tau}^{\zeta_\tau} \text{sgn}(Y_s) \{(\bar{g}_c(s, Y_s) - g(s, 0)) dA_s^\mu + \bar{g}(s, 0) dA_s\} - \int_{t \wedge \zeta_\tau}^{\zeta_\tau} \text{sgn}(Y_{s-}) dM_s. \end{aligned}$$

From this, (3.15) and (P3), (P5) we get

$$|Y_t| = E_z(|Y_t| | \mathcal{F}_t) \leq E_z\left(|\xi| + \int_0^{\zeta_\tau} (|f(s, 0)| ds + |g(s, 0)| dA_s) | \mathcal{F}_t\right) \leq c, \quad (3.16)$$

which shows that in fact  $(Y, M)$  is a solution of (3.15) with  $\bar{f}_c$  replaced by  $f$  and  $\bar{g}_c$  replaced by  $g$ .

Step 2. For  $n \geq 0$  set  $\xi^n = T_n(\xi)$ ,  $f_n(t, y) = f(t, y) - f(t, 0) + T_n(f(t, 0))$  (with  $\xi, f(t, y)$  defined in Step 1) and  $A_t^n = \int_0^t \mathbf{1}_{\{A_s^\mu \leq n\}} dA_s^\mu$ . By Step 1, for each  $n \geq 0$  there exists a unique solution  $(Y^n, M^n)$  of the BSDE

$$\begin{aligned} Y_t^n &= \xi^n + \int_{t \wedge \zeta_\tau}^{\zeta_\tau} f_n(s, Y_s^n) ds + \int_{t \wedge \zeta_\tau}^{\zeta_\tau} (g(s, Y_s^n) - g(s, 0)) dA_s^\mu \\ &\quad + \int_{t \wedge \zeta_\tau}^{\zeta_\tau} T_n g(s, 0) dA_s^n - \int_{t \wedge \zeta_\tau}^{\zeta_\tau} dM_s^n, \quad t \geq 0 \end{aligned} \quad (3.17)$$

on the space  $(\Omega, \mathcal{F}, P_z)$  (as in Step 1, for brevity, in notation we drop the dependence of  $(Y^n, M^n)$  on  $z$ ). For  $m \geq n \geq 0$  write  $\delta Y = Y^m - Y^n$ ,  $\delta M = M^m - M^n$ ,  $\delta \xi = \xi^m - \xi^n$ . Since  $\mu \in \mathcal{R}^+(E_{0,T})$ ,  $A^m$  is an increasing process. Therefore using the Meyer-Tanaka formula we obtain

$$\begin{aligned} |\delta Y_t| &\leq |\delta \xi| + \int_{t \wedge \zeta_\tau}^{\zeta_\tau} \text{sgn}(\delta Y_s) (f_m(s, Y_s^m) - f_n(s, Y_s^n)) ds \\ &\quad + \int_{t \wedge \zeta_\tau}^{\zeta_\tau} \text{sgn}(\delta Y_s) (g(s, Y_s^m) - g(s, Y_s^n)) dA_s^\mu \\ &\quad + \int_{t \wedge \zeta_\tau}^{\zeta_\tau} \text{sgn}(\delta Y_s) \{T_m g(s, 0) dA_s^m - T_n g(s, 0) dA_s^n\} + \int_{t \wedge \zeta_\tau}^{\zeta_\tau} \text{sgn}(\delta Y_{s-}) d(\delta M)_s. \end{aligned}$$

From the above and (P3), (P5) it follows that

$$\begin{aligned} |\delta Y_t| &\leq |\delta \xi| + \int_{t \wedge \zeta_\tau}^{\zeta_\tau} |T_m f(s, 0) - T_n f(s, 0)| ds + \int_{t \wedge \zeta_\tau}^{\zeta_\tau} |T_m g(s, 0) - T_n g(s, 0)| dA_s^m \\ &\quad + \int_{t \wedge \zeta_\tau}^{\zeta_\tau} |T_n g(s, 0)| d(A_s^m - A_s^n) + \int_{t \wedge \zeta_\tau}^{\zeta_\tau} \text{sgn}(\delta Y_{s-}) d(\delta M)_s. \end{aligned}$$

Hence,

$$|\delta Y_t| = E_z(|\delta Y_t| | \mathcal{F}_t) \leq E_z(\Psi^n | \mathcal{F}_t), \quad t \geq 0, \quad (3.18)$$

where

$$\begin{aligned}\Psi^n &= |\xi| \mathbf{1}_{\{|\xi| > n\}} + \int_0^{\zeta_\tau} |f(t, 0)| \mathbf{1}_{\{|f(t, 0)| > n\}} dt \\ &\quad + \int_0^{\zeta_\tau} |g(t, 0)| \mathbf{1}_{\{|g(t, 0)| > n\}} dA_t^\mu + \int_0^{\zeta_\tau} |g(t, 0)| d(A_t^m - A_t^n).\end{aligned}$$

Observe that from our assumptions on the data  $\varphi, f, g, \mu$  it follows that  $E_z \Psi^n \rightarrow 0$  as  $n \rightarrow \infty$  for q.e.  $z \in E_{0,T}$ . By (3.18),  $\|\delta Y\|_{1,z} \leq E_z \Psi^n$ , while by [3, Lemma 6.1],  $E_z \sup_{t \leq T} |\delta Y_t|^q \leq (1-q)^{-1} (E_z \Psi^n)^q$  for every  $q \in (0, 1)$ . Since the spaces  $\mathcal{D}^q(P_z)$  and  $\mathcal{L}^1(P_z)$  are complete, for q.e.  $z \in E_{0,T}$  there exists a process  $Y^z$  such that  $Y^z \in \mathcal{D}^q(P_z)$  for  $q \in (0, 1)$ ,  $Y^z$  is of class D under  $P_z$  and

$$\|Y^n - Y^z\|_{1,z} \rightarrow 0, \quad E_z \sup_{0 \leq t \leq \zeta_\tau} |Y_t^n - Y_t^z|^q \rightarrow 0. \quad (3.19)$$

We have

$$\begin{aligned}\int_0^{\zeta_\tau} |f_n(t, Y_t^n) - f(t, Y_t^z)| dt &\leq \int_0^{\zeta_\tau} |f(t, Y_t^n) - f(t, Y_t^z)| dt \\ &\quad + \int_0^{\zeta_\tau} |f(t, 0)| \mathbf{1}_{\{|f(t, 0)| > n\}} dt.\end{aligned}$$

Applying the Meyer-Tanaka formula we get (see the proof of (3.16))

$$|Y_t^n| \leq E_z \left( |\xi| + \int_0^{\zeta_\tau} |f(s, 0)| ds + \int_0^{\zeta_\tau} |g(s, 0)| dA_s^\mu \middle| \mathcal{F}_t \right) := R_t, \quad t \geq 0.$$

For  $k, N \in \mathbb{N}$  set

$$\begin{aligned}\tau_{k,N} &= \inf \{ t \geq 0 : R_t \geq k, \int_0^t (|f(s, -k)| + |f(s, k)|) ds \\ &\quad + \int_0^t (|g(s, -k)| + |g(s, k)|) dA_s^\mu \geq N \} \wedge \zeta_\tau.\end{aligned}$$

By (3.17),

$$\begin{aligned}Y_{t \wedge \tau_{k,N}}^n &= E_z \left( Y_{\tau_{k,N}}^n + \int_{t \wedge \tau_{k,N}}^{\tau_{k,N}} f_n(s, Y_s^n) ds \right. \\ &\quad \left. + \int_{t \wedge \tau_{k,N}}^{\tau_{k,N}} \{ g(s, Y_s^n) - g(s, 0) \} dA_s^\mu + T_n g(s, 0) dA_s^n \middle| \mathcal{F}_t \right).\end{aligned} \quad (3.20)$$

From the definition of  $\tau_{k,N}$  it follows that

$$\int_0^{\tau_{k,N}} |f(t, Y_t^n)| dt + \int_0^{\tau_{k,N}} |g(t, Y_t^n)| dA_t^\mu \leq N.$$

From this, (P2) and (3.19) one can deduce that

$$\lim_{n \rightarrow \infty} E_z \int_0^{\tau_{k,N}} (|f_n(t, Y_t^n) - f(t, Y_t^z)|) dt = 0 \quad (3.21)$$

and

$$\lim_{n \rightarrow \infty} E_z \int_0^{\tau_{k,N}} \{|g(t, Y_t^n) - g(t, 0)| dA_t^\mu + |T_n g(t, 0)| dA_t^n\} = 0. \quad (3.22)$$

By Doob's inequality (see, e.g., [20, Theorem 1.9.1]) and (3.19), for every  $\varepsilon > 0$  we have

$$\lim_{n \rightarrow \infty} P_x \left( \sup_{t \leq T} |E_z(Y_{\tau_{k,N}}^n - Y_{\tau_{k,N}}^z | \mathcal{F}_t)| > \varepsilon \right) \leq \varepsilon^{-1} \lim_{n \rightarrow \infty} E_z |Y_{\tau_{k,N}}^n - Y_{\tau_{k,N}}^z| = 0. \quad (3.23)$$

Similarly, by (3.21), (3.22) and Doob's inequality,

$$\lim_{n \rightarrow \infty} P_z \left( \sup_{t \leq T} |E_z \left( \int_{t \wedge \tau_{k,N}}^{\tau_{k,N}} (f(s, Y_s^n) - f(s, Y_s^z)) ds | \mathcal{F}_t \right)| > \varepsilon \right) = 0 \quad (3.24)$$

and

$$\lim_{n \rightarrow \infty} P_z \left( \sup_{t \leq T} |E_z \left( \int_{t \wedge \tau_{k,N}}^{\tau_{k,N}} (g(s, Y_s^n) - g(s, Y_s^z)) dA_s^\mu | \mathcal{F}_t \right)| > \varepsilon \right) = 0 \quad (3.25)$$

for every  $\varepsilon > 0$ . Letting  $n \rightarrow \infty$  in (3.20) and using (3.23)–(3.25) we conclude that

$$Y_{t \wedge \tau_{k,N}}^z = E_z \left( Y_{\tau_{k,N}}^z + \int_{t \wedge \tau_{k,N}}^{\tau_{k,N}} \{f(s, Y_s^z) ds + g(s, Y_s^z) dA_s^\mu\} | \mathcal{F}_t \right). \quad (3.26)$$

We have

$$\begin{aligned} & \int_0^{\zeta_\tau} |f_n(t, Y_t^n)| dt + \int_0^{\zeta_\tau} |g(t, Y_t^n)| dA_t^\mu \\ & \leq \int_0^{\zeta_\tau} |f_n(t, Y_t^n) - f_n(t, 0)| dt + \int_0^{\zeta_\tau} |f_n(t, 0)| dt \\ & \quad + \int_0^{\zeta_\tau} |g(t, Y_t^n) - g(t, 0)| dA_t^\mu + \int_0^{\zeta_\tau} |g(t, 0)| dA_t^\mu \\ & = - \int_0^{\zeta_\tau} \text{sgn}(Y_t^n) (f_n(t, Y_t^n) - f_n(t, 0)) dt + \int_0^{\zeta_\tau} |f_n(t, 0)| dt \\ & \quad - \int_0^{\zeta_\tau} \text{sgn}(Y_t^n) (g(t, Y_t^n) - g(t, 0)) dA_t^\mu + \int_0^{\zeta_\tau} |g(t, 0)| dA_t^\mu. \end{aligned}$$

But the Meyer-Tanaka formula and (3.17),

$$\begin{aligned} |\xi^n| - |Y_0^n| & \geq - \int_0^{\zeta_\tau} \text{sgn}(Y_t^n) f_n(t, Y_t^n) dt - \int_0^{\zeta_\tau} \text{sgn}(Y_t^n) g(t, Y_t^n) dA_t^\mu \\ & \quad - \int_0^{\zeta_\tau} \text{sgn}(Y_t^n) T_n g(t, 0) dA_t^n - \int_0^{\zeta_\tau} \text{sgn}(Y_{t-}^n) dM_t. \end{aligned}$$

Hence,

$$\begin{aligned} & E_z \int_0^{\zeta_\tau} \{|f_n(t, Y_t^n)| dt + |g(t, Y_t^n)| dA_t^\mu\} \\ & \leq E_z \left( |\xi^n| + 2 \int_0^{\zeta_\tau} |f(t, 0)| dt + 3 \int_0^{\zeta_\tau} |g(t, 0)| dA_t^\mu \right), \end{aligned}$$

so applying Fatou's lemma and (3.19) gives

$$\begin{aligned} E_z \int_0^{\zeta_\tau} \{|f(t, Y_t^z)| dt + |g(t, Y_t^z)| dA_t^\mu\} \\ \leq E_z \left( \varphi(\mathbf{X}_{\zeta_\tau}) + 2 \int_0^{\zeta_\tau} |f(t, 0)| dt + 3 \int_0^{\zeta_\tau} |g(t, 0)| dA_t^\mu \right) < \infty. \end{aligned} \quad (3.27)$$

Since  $f(\cdot, -k), f(\cdot, k) \in qL^1(E_{0,T}; m_1)$  and  $g(\cdot, -k), g(\cdot, k) \in qL^1(E_{0,T}; \mu)$ ,  $\tau_{k,N} \rightarrow \tau_k$  as  $N \rightarrow \infty$ , where

$$\tau_k = \inf\{t \geq 0 : R_t \geq k\} \wedge \zeta_\tau.$$

Hence  $Y_{\tau_{k,N}}^z \rightarrow Y_{\tau_k}^z$ ,  $P_z$ -a.s., and consequently,

$$\lim_{N \rightarrow \infty} E_z |Y_{\tau_{k,N}}^z - Y_{\tau_k}^z| = 0 \quad (3.28)$$

since  $Y^z$  is of class D. Letting  $N \rightarrow \infty$  in (3.26) and using (3.27), (3.28) and Doob's inequality we obtain

$$Y_t^z = E_z \left( Y_{\tau_k}^z + \int_{t \wedge \tau_k}^{\tau_k} \{f(s, Y_s^z) ds + g(s, Y_s^z) dA_s^\mu\} \middle| \mathcal{F}_t \right). \quad (3.29)$$

Since  $\tau_k \rightarrow \zeta_\tau$  as  $k \rightarrow \infty$ , letting  $k \rightarrow \infty$  in (3.29) and repeating arguments used to prove (3.29) we get

$$Y_t^z = E_z \left( \xi + \int_{t \wedge \zeta_\tau}^{\zeta_\tau} \{f(s, Y_s^z) ds + g(s, Y_s^z) dA_s^\mu\} \middle| \mathcal{F}_t \right).$$

We may now repeat the reasoning following Eq. (3.6) in [17] with the process  $V$  from [17] replaced by  $\int_0^\cdot g(t, Y_t^z) dA_t^\mu$  (see also the reasoning following (4.26) in the present paper) to prove that the pair  $(Y^z, \tilde{M}^z)$ , where  $\tilde{M}^z$  is a càdlàg version of the martingale

$$t \mapsto E_z \left( \varphi(\mathbf{X}_{\zeta_\tau}) + \int_0^{\zeta_\tau} \{f(\mathbf{X}_s, Y_s^z) ds + g(\mathbf{X}_s, Y_s^z) dA_s^\mu\} \middle| \mathcal{F}_t \right) - \bar{u}(\mathbf{X}_0),$$

is a solution of the BSDE

$$Y_t^z = \varphi(\mathbf{X}_{\zeta_\tau}) + \int_{t \wedge \zeta_\tau}^{\zeta_\tau} \{f(\mathbf{X}_s, Y_s^z) ds + g(\mathbf{X}_s, Y_s^z) dA_s^\mu\} - \int_{t \wedge \zeta_\tau}^{\zeta_\tau} d\tilde{M}_s^z, \quad t \geq 0 \quad (3.30)$$

on  $(\Omega, \mathcal{F}, P_z)$ . Furthermore, by [14, Remark 3.6], there exists a pair of processes  $(Y, M)$  such that  $(Y_t, M_t) = (Y^z, \tilde{M}_t^z)$ ,  $t \in [0, \zeta_\tau]$ ,  $P_z$ -a.s. for q.e.  $z \in E_{0,T}$ . Let  $u(z) = E_z Y_0$ . Then the argument from the beginning of the proof of [13, Theorem 5.8] shows that  $Y_t = u(\mathbf{X}_t)$ ,  $t \in [0, \zeta_\tau]$ , which implies that  $M$  is a version of the martingale  $M^z$  and that  $(u(\mathbf{X}), M)$  is a solution of (3.30) for q.e.  $z \in E_{0,T}$ . In view of our convention made at the beginning of Section 2.2, this means that  $(u(\mathbf{X}), M)$  is a solution of (3.4) on the space  $(\Omega, \mathcal{F}, P_z)$  for q.e.  $z \in E_{0,T}$ . Of course,  $u(\mathbf{X}) \in \mathcal{D}^q(P_z)$ . Furthermore,  $M$  is a uniformly integrable martingale under  $P_z$ , because under  $P_z$  it is a version of the closed martingale  $M^z$ . Finally, since we know that  $Y^z = u(\mathbf{X})$ ,  $t \in [0, \zeta_\tau]$ ,  $P_z$ -a.s., (3.13) follows immediately from (3.27). This completes the proof of part (ii) of the theorem.

Part (i) follows from (ii). Indeed, since  $\mu \in \mathcal{R}(E_{0,T})$  and we know that (3.27) is satisfied with  $Y^z$  replaced by  $u(\mathbf{X})$  and  $M$  is a martingale under  $P_z$  for q.e.  $z \in E_{0,T}$ , putting  $t = 0$  in (3.4) and then taking the expectation shows that  $\bar{u}$  is a solution of (3.2). To show that  $\bar{u}$  is unique we may argue as in the proof of [13, Theorem 5.8].  $\square$

**Remark 3.3.** If  $g$  does not depend on the last variable  $y$  then in Theorem 3.2 we may replace the assumptions  $\mu \in \mathcal{R}^+(E_{0,T})$ ,  $g(\cdot, \cdot, 0) \cdot \mu \in \mathcal{R}(E_{0,T})$  by the assumption  $g \cdot \mu \in \mathcal{R}(E_{0,T})$  (see [13, Theorem 5.8]).

**Remark 3.4.** (i) By [13, Proposition 3.4], the solution  $u$  of Theorem 3.2 is quasi-continuous.

(ii) Let the assumptions of Theorem 3.2 hold, and moreover,  $f(\cdot, \cdot, 0) \in L^1(E_{0,T}; m_1)$ ,  $g(\cdot, \cdot, 0) \cdot \mu \in \mathcal{M}_{0,b}(E_{0,T})$  and for some  $\gamma \geq \alpha_0$  the form  $\mathcal{E}_\gamma^{0,T}$  has the dual Markov property (for the definition of  $\mathcal{E}^{0,T}$  see [13, Section 3.3]). Then by [13, Proposition 3.13] and (3.13),

$$\begin{aligned} \|f_u\|_{L^1(E_{0,T}; m_1)} + \|g_u \cdot \mu\|_{TV} &\leq c(\|\varphi\|_{L^1(E; m)} + \|f(\cdot, \cdot, 0)\|_{L^1(E_{0,T}; m_1)} \\ &\quad + \|g(\cdot, \cdot, 0) \cdot \mu\|_{TV}). \end{aligned}$$

Therefore, by [13, Theorem 3.12],  $u \in L^1(E_{0,T}; m_1)$ ,  $T_k u \in L^2(0, T; V)$  for  $k > 0$  ( $T_k u$  is defined by (3.14)) and for every  $k > 0$  there is  $C > 0$  depending only on  $k, \alpha, T$  such that

$$\int_0^T B^{(t)}(T_k \bar{u}(t), T_k \bar{u}(t)) dt \leq C(\|\varphi\|_{L^1(E; m)} + \|f(\cdot, \cdot, 0)\|_{L^1(E_{0,T}; m_1)} + \|g(\cdot, \cdot, 0) \cdot \mu\|_{TV}),$$

where  $\|\cdot\|_{TV}$  denotes the total variation norm. Moreover, if the forms  $(B^{(t)}, V)$  are (non-symmetric) Dirichlet forms, then by [16, Theorem 4.5],  $u$  is a renormalized solution of (3.2) in the sense defined in [16].

**Remark 3.5.** In Theorem 3.2 we have assumed that the AF  $A^\mu$  is continuous. In the general case where  $\mu \in \mathcal{R}^+(E_{0,T})$  and  $A^\mu$  is possibly discontinuous, one can prove the existence of a solution of (3.2) in the following sense: there exists  $u : E_T \rightarrow \mathbb{R}$  such that  $f_u \cdot m, g_u \cdot \mu \in \mathcal{R}(E_{0,T})$  and (3.5) is satisfied with  $g_u$  replaced by  $g_{\hat{u}}$ , where  $\hat{u}$  is the precise version of  $u$  (for the notion of a precise version of a parabolic potential see [30]). In the paper we decided to provide the proof of less general result, because it suffices for the purposes of Sections 4–6 in which our main results are proved, and on the other hand, the proof of the general result is more technical than the proof of Theorem 3.2. Also note that by [13, Proposition 3.4] the solution  $u$  described above is quasi-càdlàg.

## 4 Convergence of BSDEs and Elliptic PDEs

In this section we assume that (2.9). By  $L$  we denote the operator associated via (3.1) with the form  $(B, V)$ . We also assume that  $\mu \in \mathcal{R}^+(E_{0,T})$  does not depend on time and  $f, g : E \times \mathbb{R} \rightarrow \mathbb{R}$ , i.e.  $f, g$  also do not depend on time. To shorten notation, in what follows we denote  $P_{0,x}$  by  $P_x$  and  $E_{0,x}$  by  $E_x$ . Under the measure  $P_x$ ,

$$\mathbf{X}_t = (t, X_t), \quad t \geq 0, \quad \zeta_\tau = T \wedge \zeta. \quad (4.1)$$

and

$$A_t^\mu = A_t^{0, \tilde{\mu}}, \quad t \geq 0,$$

where  $\tilde{\mu}$  is determined by (2.6).

In the rest of the paper we say that some property is satisfied quasi-everywhere (q.e. for brevity) if the set of those  $x \in E$  for which it does not hold is exceptional with respect to the form  $(B, V)$ .

Let  $\nu \in S(E)$ . We will say that a Borel measurable  $F : E \rightarrow \mathbb{R}$  is  $\nu$ -quasi-integrable ( $F \in qL^1(E; \nu)$  in notation) if for every  $T > 0$ ,  $P_x \int_0^{\zeta \wedge T} |F(X_t)| dA_t^{0, \nu} < \infty = 1$  for q.e.  $x \in E$ .

Note that in case  $\nu = m$  the notion of quasi-integrability was introduced in [12, Section 2]. For a comparison of the notion of  $m$ -integrability and the notion of quasi-integrability in the analytic sense see [12, Remark 2.3].

In this section and Section 5 we will assume that the data satisfy the following conditions.

- (E1)  $\varphi \in L^1(E; m)$ ,  $\tilde{\mu} \in \mathcal{R}^+(E)$ .
- (E2)  $f(\cdot, y), g(\cdot, y)$  are measurable for every  $y \in \mathbb{R}$  and  $f(x, \cdot), g(x, \cdot)$  are continuous for every  $x \in E$ .
- (E3)  $\langle f(x, y) - f(x, y'), y - y' \rangle \leq 0$  for all  $y, y' \in \mathbb{R}$  and  $x \in E$ .
- (E4)  $f(\cdot, 0) \cdot m \in \mathcal{R}(E)$  and  $f(\cdot, y) \in qL^1(E; m)$  for every  $y \in \mathbb{R}$ .
- (E5)  $\langle g(x, y) - g(x, y'), y - y' \rangle \leq 0$  for all  $y, y' \in \mathbb{R}$  and  $x \in E$ .
- (E6)  $g(\cdot, 0) \cdot \tilde{\mu} \in \mathcal{R}(E)$  and  $g(\cdot, y) \in qL^1(E; \tilde{\mu})$  for every  $y \in \mathbb{R}$ .

**Definition.** Let  $x \in E$ . We say that a pair  $(Y^x, M^x)$  is a solution of the BSDE

$$Y_t^x = \int_{t \wedge \zeta}^{\zeta} f(X_s, Y_s^x) ds + \int_{t \wedge \zeta}^{\zeta} g(X_s, Y_s^x) dA_s^\mu - \int_{t \wedge \zeta}^{\zeta} dM_s^x, \quad t \geq 0 \quad (4.2)$$

on the space  $(\Omega, \mathcal{F}, P_x)$  if

- (a)  $Y^x$  is an  $(\mathcal{F}_t)$ -progressively measurable process of class D under  $P_x$ ,  $Y_{t \wedge \zeta}^x \rightarrow 0$ ,  $P_x$ -a.s. as  $t \rightarrow \infty$  and  $M^x$  is an  $(\mathcal{F}_t)$ -local martingale under  $P_x$  such that  $M_0 = 0$ ,
- (b) For every  $T > 0$ ,  $\int_0^T |f(X_t, Y_t^x)| dt < \infty$ ,  $\int_0^T |g(X_t, Y_t^x)| d|A^\mu|_t < \infty$ ,  $P_x$ -a.s. and

$$Y_t^x = Y_{T \wedge \zeta}^x + \int_{t \wedge \zeta}^{T \wedge \zeta} f(X_s, Y_s^x) ds + \int_{t \wedge \zeta}^{T \wedge \zeta} g(X_s, Y_s^x) dA_s^\mu - \int_{t \wedge \zeta}^{T \wedge \zeta} dM_s^x, \quad t \in [0, T].$$

**Definition.** We say that  $v : E \rightarrow \mathbb{R}$  is a solution of problem (1.2) if  $f_v \cdot m \in \mathcal{R}(E)$ ,  $g_v \cdot \tilde{\mu} \in \mathcal{R}(E)$  and for q.e.  $x \in E$ ,

$$v(x) = E_x \left( \int_0^{\zeta} f_v(X_t) dt + \int_0^{\zeta} g_v(X_t) dA_t^\mu \right). \quad (4.3)$$

Suppose that for some  $x \in E$  for every  $n > 0$  there exists a solution  $(Y^n, M^n)$  of the BSDE

$$\begin{aligned} Y_t^n &= \mathbf{1}_{\{\zeta > n\}} \varphi(X_n) + \int_t^{n \wedge \zeta} f(X_s, Y_s^n) ds + \int_t^{n \wedge \zeta} g(X_s, Y_s^n) dA_s^\mu \\ &\quad - \int_t^{n \wedge \zeta} dM_s^n, \quad t \in [0, n], \quad P_x\text{-a.s.} \end{aligned} \quad (4.4)$$



on the probability space  $(\Omega, \mathcal{F}, P_x)$ . The solutions may depend on  $x$  but for brevity in notation we drop the dependence of  $Y^n, M^n$  on  $x$ . In what follows by  $\tilde{Y}^n, \tilde{M}^n$  we denote processes defined as

$$\tilde{Y}_t^n = Y_t^n, \quad \tilde{M}_t^n = M_t^n, \quad t < n, \quad \tilde{Y}_t^n = 0, \quad \tilde{M}_t^n = M_n^n, \quad t \geq n. \quad (4.5)$$

**Proposition 4.1.** *Assume that (E1)–(E6) are satisfied. For  $0 < n < m$  set  $\delta Y = \tilde{Y}^m - \tilde{Y}^n$ . Then for every  $x \in E$ ,*

$$\begin{aligned} \|\delta Y\|_{x,1} &\leq E_x \left( \mathbf{1}_{\{\zeta > m\}} |\varphi(X_m)| + \mathbf{1}_{\{\zeta > n\}} |\varphi(X_n)| \right. \\ &\quad \left. + \int_{n \wedge \zeta}^{m \wedge \zeta} |f(X_t, 0)| dt + \int_{n \wedge \zeta}^{m \wedge \zeta} |g(X_t, 0)| dA_t^\mu \right) \end{aligned} \quad (4.6)$$

and

$$\begin{aligned} E_x \sup_{t \geq 0} |\delta Y_t|^q &\leq \frac{1}{1-q} \left( E_x (\mathbf{1}_{\{\zeta > m\}} |\varphi(X_m)| + \mathbf{1}_{\{\zeta > n\}} |\varphi(X_n)|) \right. \\ &\quad \left. + E_x \int_{n \wedge \zeta}^{m \wedge \zeta} |f(X_t, 0)| dt + E_x \int_{n \wedge \zeta}^{m \wedge \zeta} |g(X_t, 0)| dA_t^\mu \right)^q \end{aligned} \quad (4.7)$$

for every  $q \in (0, 1)$ . Moreover, for every  $t \geq 0$ ,

$$\begin{aligned} E_x \int_0^{t \wedge \zeta} |f(X_s, Y_s^n)| ds + E_x \int_0^{t \wedge \zeta} |g(X_s, Y_s^n)| dA_s^\mu \\ \leq E_x \left( |Y_t^n| + 2 \int_0^{t \wedge \zeta} |f(X_s, 0)| ds + 2 \int_0^{t \wedge \zeta} |g(X_s, 0)| dA_s^\mu \right). \end{aligned} \quad (4.8)$$

*Proof.* By (4.4),

$$\begin{aligned} Y_t^n &= Y_0^n - \int_0^{t \wedge \zeta} \{f(X_s, Y_s^n) ds + g(X_s, Y_s^n) dA_s^\mu\} + \int_0^{t \wedge \zeta} dM_s^n \\ &= Y_0^n - \int_0^t (\mathbf{1}_{[0, n \wedge \zeta]}(s) f(X_s, Y_s^n) ds + \mathbf{1}_{[0, n \wedge \zeta]}(s) g(X_s, Y_s^n) dA_s^\mu) \\ &\quad + \int_0^t \mathbf{1}_{[0, n \wedge \zeta]}(s) dM_s^n, \quad t \in [0, n], \quad P_x\text{-a.s.} \end{aligned} \quad (4.9)$$

From the above and the fact that the process  $A^\mu$  is continuous it follows that the pair  $(\tilde{Y}^n, \tilde{M}^n)$  defined (4.5) satisfies

$$\begin{aligned} \tilde{Y}_t^n &= Y_0^n - \int_0^t (\mathbf{1}_{[0, n \wedge \zeta]}(s) f(X_s, \tilde{Y}_s^n) ds + \mathbf{1}_{[0, n \wedge \zeta]}(s) g(X_s, \tilde{Y}_s^n) dA_s^\mu) \\ &\quad + \int_0^t dV_s^n + \int_0^t \mathbf{1}_{[0, n \wedge \zeta]}(s) d\tilde{M}_s^n, \quad t \geq 0, \end{aligned} \quad (4.10)$$

where

$$V_t^n = 0 \text{ if } t < n, \quad V_t^n = -Y_n^n \text{ if } t \geq n.$$

Let  $\delta \tilde{Y} = \tilde{Y}^m - \tilde{Y}^n$ . By (4.10),

$$\delta \tilde{Y}_t = \delta \tilde{Y}_0 + K_t + \int_0^t (\mathbf{1}_{[0, m \wedge \zeta]}(s) d\tilde{M}_s^m - \mathbf{1}_{[0, n \wedge \zeta]}(s) d\tilde{M}_s^n), \quad t \geq 0, \quad P_x\text{-a.s.},$$

where

$$\begin{aligned} K_t = & - \int_0^t \mathbf{1}_{[0, n \wedge \zeta]}(s) (f(X_s, \tilde{Y}_s^m) - f(X_s, \tilde{Y}_s^n)) ds - \int_0^t \mathbf{1}_{(n \wedge \zeta, m \wedge \zeta]}(s) f(X_s, \tilde{Y}_s^m) ds \\ & - \int_0^t \mathbf{1}_{[0, n \wedge \zeta]}(s) (g(X_s, \tilde{Y}_s^m) - g(X_s, \tilde{Y}_s^n)) dA_s^\mu \\ & - \int_0^t \mathbf{1}_{(n \wedge \zeta, m \wedge \zeta]}(s) g(X_s, \tilde{Y}_s^m) dA_s^\mu + \int_0^t d(V_s^m - V_s^n). \end{aligned}$$

By the Meyer-Tanaka formula (see [32, p. 216]), for  $t < m$  we have

$$|\delta \tilde{Y}_m| - |\delta \tilde{Y}_t| \geq \int_t^m \operatorname{sgn}(\delta \tilde{Y}_{s-}) d(\delta \tilde{Y})_s,$$

where  $\operatorname{sgn}(x) = 1$  if  $x > 0$  and  $\operatorname{sgn}(x) = -1$  if  $x \leq 0$ . Therefore, for  $t < m$ ,

$$|\delta \tilde{Y}_t| = E_x(|\delta \tilde{Y}_t| | \mathcal{F}_t) \leq E_x \left( |\delta \tilde{Y}_m| - \int_t^m \operatorname{sgn}(\delta \tilde{Y}_{s-}) dK_s \mid \mathcal{F}_t \right).$$

From this it follows that for  $t \in [0, m]$ ,

$$\begin{aligned} |\delta \tilde{Y}_t| \leq & E_x \left( |\delta \tilde{Y}_m| + \int_t^m \mathbf{1}_{[0, n \wedge \zeta]}(s) \operatorname{sgn}(\delta \tilde{Y}_s) (f(X_s, \tilde{Y}_s^m) - f(X_s, \tilde{Y}_s^n)) ds \right. \\ & + \int_t^m \mathbf{1}_{[0, n \wedge \zeta]}(s) \operatorname{sgn}(\delta \tilde{Y}_s) (g(X_s, \tilde{Y}_s^m) - g(X_s, \tilde{Y}_s^n)) dA_s^\mu \\ & + \int_t^m \mathbf{1}_{(n \wedge \zeta, m \wedge \zeta]}(s) \operatorname{sgn}(\delta \tilde{Y}_s) f(X_s, \tilde{Y}_s^m) ds \\ & \left. + \int_t^m \mathbf{1}_{(n \wedge \zeta, m \wedge \zeta]}(s) \operatorname{sgn}(\delta \tilde{Y}_s) g(X_s, \tilde{Y}_s^m) dA_s^\mu + |V_m^m| + |V_n^n| \mid \mathcal{F}_t \right). \end{aligned}$$

By (E3),

$$\int_t^m \mathbf{1}_{[0, n \wedge \zeta]}(s) \operatorname{sgn}(\delta \tilde{Y}_s) (f(X_s, \tilde{Y}_s^m) - f(X_s, \tilde{Y}_s^n)) ds \leq 0,$$

whereas by (E5) and the fact that  $A^\mu$  is increasing,

$$\int_t^m \mathbf{1}_{[0, n \wedge \zeta]}(s) \operatorname{sgn}(\delta \tilde{Y}_s) (g(X_s, \tilde{Y}_s^m) - g(X_s, \tilde{Y}_s^n)) dA_s^\mu \leq 0.$$

Furthermore, since  $\tilde{Y}_t^n = 0$  for  $t \geq n$ , it follows from (E3) (with  $\alpha \leq 0$ ) that

$$\begin{aligned} \int_t^m \mathbf{1}_{(n \wedge \zeta, m \wedge \zeta]}(s) \operatorname{sgn}(\delta \tilde{Y}_s) f(X_s, \tilde{Y}_s^m) ds & \leq \int_t^m \mathbf{1}_{(n \wedge \zeta, m \wedge \zeta]}(s) \operatorname{sgn}(\delta \tilde{Y}_s) f(X_s, 0) ds \\ & \leq \int_{n \wedge \zeta}^{m \wedge \zeta} |f(X_s, 0)| ds. \end{aligned}$$

Similarly, by (E5),

$$\int_t^m \mathbf{1}_{(n \wedge \zeta, m \wedge \zeta]}(s) \operatorname{sgn}(\delta \tilde{Y}_s) g(X_s, \tilde{Y}_s^m) dA_s^\mu \leq \int_{n \wedge \zeta}^{m \wedge \zeta} |g(X_s, 0)| dA_s^\mu.$$

Furthermore,  $\delta\tilde{Y}_m = 0$  and

$$|V_m^m| + |V_n^n| = |Y_m^m| + |Y_n^n| = \mathbf{1}_{\{\zeta > m\}}|\varphi(X_m)| + \mathbf{1}_{\{\zeta > n\}}|\varphi(X_n)|.$$

Therefore, for  $t \in [0, m]$  we have

$$\begin{aligned} |\delta\tilde{Y}_t| &\leq E_x \left( \mathbf{1}_{\{\zeta > m\}}|\varphi(X_m)| + \mathbf{1}_{\{\zeta > n\}}|\varphi(X_n)| \right. \\ &\quad \left. + \int_{n \wedge \zeta}^{m \wedge \zeta} |f(X_s, 0)| ds + \int_{n \wedge \zeta}^{m \wedge \zeta} |g(X_s, 0)| dA_s^\mu \Big| \mathcal{F}_t \right) := N_t. \end{aligned} \quad (4.11)$$

This implies (4.6). By [3, Lemma 6.1],

$$E_x \sup_{0 \leq t \leq m} |\delta\tilde{Y}_t|^q \leq (1 - q)^{-1} (E_x N_m)^q,$$

which shows (4.7). Finally, to prove (4.8), let us first observe that by the Meyer-Tanaka formula,

$$E_x |Y_t^n| - E_x |Y_0^n| \geq E_x \int_0^t \text{sgn}(Y_{s-}^n) dY_s^n.$$

By the above inequality and (4.9), for  $t < n$  we have

$$\begin{aligned} E_x |Y_t^n| - E_x |Y_0^n| \\ \geq -E_x \int_0^t \mathbf{1}_{[0, n \wedge \zeta]}(s) \text{sgn}(Y_s^n) \{f(X_s, Y_s^n) ds + g(X_s, Y_s^n) dA_s^\mu\}. \end{aligned} \quad (4.12)$$

On the other hand, for every  $t \geq 0$ ,

$$\begin{aligned} \int_0^t |g(X_s, Y_s^n)| dA_s^\mu &\leq \int_0^t |g(X_s, Y_s^n) - g(X_s, 0)| dA_s^\mu + \int_0^t |g(X_s, 0)| dA_s^\mu \\ &= - \int_0^t \text{sgn}(Y_s^n) (g(X_s, Y_s^n) - g(X_s, 0)) dA_s^\mu + \int_0^t |g(X_s, 0)| dA_s^\mu \\ &\leq - \int_0^t \text{sgn}(Y_s^n) g(X_s, Y_s^n) dA_s^\mu + 2 \int_0^t |g(X_s, 0)| dA_s^\mu, \end{aligned}$$

and similarly,

$$\int_0^t |f(X_s, Y_s^n)| ds \leq - \int_0^t \text{sgn}(Y_s^n) f(X_s, Y_s^n) ds + 2 \int_0^t |f(X_s, 0)| ds,$$

which when combined with (4.12) proves (4.8).  $\square$

**Proposition 4.2.** *Assume that (E1)–(E6) are satisfied and*

$$\lim_{t \rightarrow \infty} E_x \mathbf{1}_{\{\zeta > t\}} |\varphi(X_t)| = 0. \quad (4.13)$$

*Assume for some  $x \in E$  for each  $n \in \mathbb{N}$  there exists a solution  $(Y^n, M^n)$  of (4.4) on the space  $(\Omega, \mathcal{F}, P_x)$ . If*

$$E_x \int_0^\zeta |f(X_t, 0)| dt + E_x \int_0^\zeta |g(X_t, 0)| dA_t^\mu < \infty \quad (4.14)$$

then there exists a solution  $(Y^x, M^x)$  of (4.2) on  $(\Omega, \mathcal{F}, P_x)$ . Moreover,  $Y^x \in \mathcal{D}^q(P_x)$  for  $q \in (0, 1)$ ,  $M^x$  is a uniformly integrable  $(\mathcal{F}_t)$ -martingale under  $P_x$  and

$$\begin{aligned} E_x \int_0^\zeta |f(X_t, Y_t^x)| dt + E_x \int_0^\zeta |g(X_t, Y_t^x)| dA_t^\mu \\ \leq 2E_x \left( \int_0^\zeta |f(X_t, 0)| dt + \int_0^\zeta |g(X_t, 0)| dA_t^\mu \right). \end{aligned} \quad (4.15)$$

Finally,

$$\lim_{n \rightarrow \infty} \|Y^n - Y^x\|_{x,1} = 0 \quad (4.16)$$

and for every  $q \in (0, 1)$ ,

$$\lim_{n \rightarrow \infty} E_x \sup_{t \geq 0} |Y_t^n - Y_t^x|^q = 0. \quad (4.17)$$

*Proof.* From (4.6) and (4.13), (4.14) it follows that for every  $x \in E$ ,  $\|Y^n - Y^m\|_{x,1} \rightarrow 0$  as  $n, m \rightarrow \infty$ . Hence there exists a process  $Y \in \mathcal{L}^1(P_x)$  of class D such that (4.16) is satisfied. By (4.7), (4.13) and (4.14),  $\lim_{n, m \rightarrow \infty} E_x \sup_{t \geq 0} |Y_t^n - Y_t^m|^q \rightarrow 0$ . Since the space  $\mathcal{D}^q(P_x)$  is complete, the last convergence and (4.16) imply that  $Y^x \in \mathcal{D}^q(P_x)$  and (4.17) is satisfied. From (4.8), (4.16), (4.17) and Fatou's lemma it follows that for every  $T > 0$ ,

$$\begin{aligned} E_x \int_0^{T \wedge \zeta} |f(X_t, Y_t^x)| dt + E_x \int_0^{T \wedge \zeta} |g(X_t, Y_t^x)| dA_t^\mu \\ \leq 2E_x \left( |Y_{T \wedge \zeta}^x| + \int_0^{T \wedge \zeta} |f(X_t, 0)| ds + \int_0^{T \wedge \zeta} |g(X_t, 0)| dA_t^\mu \right). \end{aligned}$$

Since  $Y_\zeta^n = 0$ ,  $P_x$ -a.s. for  $n \in \mathbb{N}$ , from (4.17) we conclude that  $Y_{T \wedge \zeta}^x \rightarrow 0$  in probability  $P_x$  as  $T \rightarrow \infty$ . As a consequence, since  $Y^x$  is of class D,  $E_x |Y_{T \wedge \zeta}^x| \rightarrow 0$ . Therefore letting  $T \rightarrow \infty$  in the last inequality we get (4.15). Using (4.17) one can show that  $\int_0^\zeta |g(X_t, Y_t^n) - g(X_t, Y_t^x)| dA_t^\mu \rightarrow 0$  in probability  $P_x$  (see the proof of [11, Eq. (6.16)]). Set  $F_R(t, x) = |f(t, x, -R)| \vee |f(t, x, R)|$ ,  $G_R(t, x) = |g(t, x, -R)| \vee |g(t, x, R)|$  and for  $N, R > 0$  and  $n \in \mathbb{N}$  define the stopping times

$$\tau_{n,R} = \inf\{t \geq 0 : |Y_t^n| > R\}, \quad \tau_R = \inf_{n \geq R} \tau_{n,R}$$

and

$$\sigma_{N,R} = \inf\{t \geq 0 : \int_0^t (F_R(X_s) ds + G_R(X_s) dA_s^\mu) > N\}, \quad \delta_{N,R} = \sigma_{N,R} \wedge \tau_R.$$

By (4.4), for  $T < n$  we have

$$\begin{aligned} Y_{t \wedge \zeta \wedge \delta_{N,R}}^n &= Y_{T \wedge \zeta \wedge \delta_{N,R}}^n + \int_{t \wedge \zeta \wedge \delta_{N,R}}^{T \wedge \zeta \wedge \delta_{N,R}} \{f(X_s, Y_s^n) ds + g(X_s, Y_s^n) dA_s^\mu\} \\ &\quad - \int_{t \wedge \zeta \wedge \delta_{N,R}}^{T \wedge \zeta \wedge \delta_{N,R}} dM_s^n, \quad t \in [0, T], \quad P_x\text{-a.s.} \end{aligned}$$

Since  $Y_t^n = Y_{t \wedge \zeta}^n$  and  $\int_{t \wedge \zeta \wedge \delta_{N,R}}^{T \wedge \zeta \wedge \delta_{N,R}} dM_s^n = \int_t^T dM_{s \wedge \zeta \wedge \delta_{N,R}}^n$  and the martingale  $M^n$  stopped at  $\zeta \wedge \delta_{N,R}$  is still a martingale (see [32, Theorem I.18]), it follows that

$$Y_{t \wedge \zeta \wedge \delta_{N,R}}^n = E_x \left( Y_{T \wedge \zeta \wedge \delta_{N,R}}^n + \int_{t \wedge \zeta \wedge \delta_{N,R}}^{T \wedge \zeta \wedge \delta_{N,R}} \{f(X_s, Y_s^n) ds + g(X_s, Y_s^n) dA_s^\mu\} | \mathcal{F}_t \right). \quad (4.18)$$

By Doob's inequality (see, e.g., [20, Theorem 1.9.1]) and (4.16) for every  $\varepsilon > 0$  we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} P_x \left( \sup_{t \leq T} |E_x(Y_{T \wedge \zeta \wedge \delta_{N,R}}^n - Y_{T \wedge \zeta \wedge \delta_{N,R}}^x | \mathcal{F}_t)| > \varepsilon \right) \\ & \leq \varepsilon^{-1} \lim_{n \rightarrow \infty} E_x |Y_{T \wedge \zeta \wedge \delta_{N,R}}^n - Y_{T \wedge \zeta \wedge \delta_{N,R}}^x| = 0. \end{aligned} \quad (4.19)$$

From the definition of  $\delta_{N,R}$  and (E2), (4.17) it follows that

$$\lim_{n \rightarrow \infty} E_x \int_0^{T \wedge \zeta \wedge \delta_{N,R}} \{|f(X_s, Y_s^n) - f(X_s, Y_s^x)| ds + |g(X_s, Y_s^n) - g(X_s, Y_s^x)| dA_s^\mu\} = 0.$$

Hence, by Doob's inequality (see, e.g., [20, Theorem 1.9.1]),

$$\lim_{n \rightarrow \infty} P_x \left( \sup_{t \leq T} |E_x \left( \int_{t \wedge \zeta \wedge \delta_{N,R}}^{T \wedge \zeta \wedge \delta_{N,R}} (f(X_s, Y_s^n) - f(X_s, Y_s^x)) ds | \mathcal{F}_t \right)| > \varepsilon \right) = 0 \quad (4.20)$$

and

$$\lim_{n \rightarrow \infty} P_x \left( \sup_{t \leq T} |E_x \left( \int_{t \wedge \zeta \wedge \delta_{N,R}}^{T \wedge \zeta \wedge \delta_{N,R}} (g(X_s, Y_s^n) - g(X_s, Y_s^x)) dA_s^\mu | \mathcal{F}_t \right)| > \varepsilon \right) = 0 \quad (4.21)$$

for every  $\varepsilon > 0$ . Letting  $n \rightarrow \infty$  in (4.18) and using (4.17) and (4.19)–(4.21) we conclude that  $P_x$ -a.s.

$$Y_{t \wedge \delta_{N,R}}^x = E_x \left( Y_{T \wedge \zeta \wedge \delta_{N,R}}^x + \int_{t \wedge \zeta \wedge \delta_{N,R}}^{T \wedge \zeta \wedge \delta_{N,R}} \{f(X_s, Y_s^x) ds + g(X_s, Y_s^x) dA_s^\mu\} | \mathcal{F}_t \right) \quad (4.22)$$

for  $t \in [0, T]$ . By (E4),  $F_R \in qL^1(E_{0,T}; m_1)$ , and by (E6), and  $G_R \in qL^1(E_{0,T}; \mu)$ . Therefore  $\sigma_{N,R} \nearrow \tau_R$ ,  $P_x$ -a.s. as  $N \rightarrow \infty$  for each fixed  $R > 0$ . Hence  $Y_{T \wedge \zeta \wedge \delta_{N,R}}^x \rightarrow Y_{T \wedge \zeta \wedge \tau_R}^x$ ,  $P_x$ -a.s. as  $N \rightarrow \infty$ , and consequently  $E_x |Y_{T \wedge \zeta \wedge \delta_{N,R}}^x - Y_{T \wedge \zeta \wedge \tau_R}^x| \rightarrow 0$  since  $Y^x$  is of class D. From the last convergence and Doob's inequality it follows that for every  $\varepsilon > 0$ ,

$$\lim_{N \rightarrow \infty} P_x \left( \sup_{t \leq T} |E_x(Y_{T \wedge \zeta \wedge \delta_{N,R}}^x - Y_{T \wedge \zeta \wedge \tau_R}^x | \mathcal{F}_t)| > \varepsilon \right) = 0.$$

Therefore letting  $N \rightarrow \infty$  in (4.22) and using (4.15) we show that  $P_x$ -a.s.,

$$Y_t^x = E_x \left( Y_{T \wedge \zeta \wedge \tau_R}^x + \int_{t \wedge \zeta \wedge \tau_R}^{T \wedge \zeta \wedge \tau_R} \{f(X_s, Y_s^x) ds + g(X_s, Y_s^x) dA_s^\mu\} | \mathcal{F}_t \right), \quad t \in [0, T]. \quad (4.23)$$

We now show that  $\tau_R \nearrow \infty$ ,  $P_x$ -a.s. as  $R \rightarrow \infty$ . To see this, let us suppose that  $P_x(\sup_{R>0} \tau_R \leq M) > \varepsilon$  for some  $M, \varepsilon > 0$ . Then

$$P_x(\forall_{R>0} \sup_{n \geq R} \sup_{t \leq M} |Y_t^n| \geq R) > \varepsilon. \quad (4.24)$$

Clearly,

$$\begin{aligned}
P_x(\forall_{R>0} \sup_{n \geq R} \sup_{t \leq M} |Y_t^n| \geq R) &\leq P_x(\forall_{R>0} \sup_{n \geq R} \sup_{t \leq M} |Y_t^n - Y_t| \geq R/2) \\
&\quad + P_x(\forall_{R>0} \sup_{t \leq M} |Y_t| \geq R/2) \\
&= P(\forall_{R>0} \sup_{n \geq R} \sup_{t \leq M} |Y_t^n - Y_t| \geq R/2). \tag{4.25}
\end{aligned}$$

By (4.17), taking a subsequence if necessary, we may assume that  $\sup_{t \leq M} |Y_t^n - Y_t| \rightarrow 0$ ,  $P_x$ -a.s. Therefore the random variable  $Z = \sup_{n \geq 0} \sup_{t \leq M} |Y_t^n - Y_t|$  is finite a.s., which when combined with (4.25) contradicts (4.24). This proves that  $\tau_R \nearrow \infty$ ,  $P_x$ -a.s. Now, letting  $R \rightarrow \infty$  and repeating argument used to prove (4.23), we get (4.23) with  $T \wedge \zeta \wedge \tau_R$  replaced by  $T \wedge \zeta$ . Since we know that  $E_x|Y_{T \wedge \zeta}^x| \rightarrow 0$  as  $T \rightarrow \infty$ , letting  $T \rightarrow \infty$  in this equation (i.e. in (4.23) with  $T \wedge \zeta$ ) and repeating once again the argument used to prove (4.23) we get

$$Y_t^x = E_x \left( \int_{t \wedge \zeta}^{\zeta} \{f(X_s, Y_s^x) ds + g(X_s, Y_s^x) dA_s^\mu\} | \mathcal{F}_t \right), \quad t \geq 0, \quad P_x\text{-a.s.} \tag{4.26}$$

Hence,

$$Y_t^x = \int_{t \wedge \zeta}^{\zeta} f(X_s, Y_s^x) ds + \int_{t \wedge \zeta}^{\zeta} g(X_s, Y_s^x) dA_s^\mu - \int_{t \wedge \zeta}^{\zeta} dM_s^x, \quad t \geq 0, \quad P_x\text{-a.s.}, \tag{4.27}$$

where  $M^x$  is a càdlàg version of the martingale

$$t \mapsto E_x \left( \int_0^{\zeta} f(X_s, Y_s^x) ds + \int_0^{\zeta} g(X_s, Y_s^x) dA_s^\mu | \mathcal{F}_t \right) - Y_0^x. \tag{4.28}$$

Indeed, by (4.26),

$$\begin{aligned}
Y_t^x &= E_x \left( \int_0^{\zeta} f(X_s, Y_s^x) ds + \int_0^{\zeta} g(X_s, Y_s^x) dA_s^\mu | \mathcal{F}_t \right) \\
&\quad - \int_0^{t \wedge \zeta} f(X_s, Y_s^x) ds + \int_0^{t \wedge \zeta} g(X_s, Y_s^x) dA_s^\mu, \quad t \geq 0,
\end{aligned}$$

that is

$$Y_t^x = Y_0^x + M_t^x - \int_0^{t \wedge \zeta} f(X_s, Y_s^x) ds - \int_0^{t \wedge \zeta} g(X_s, Y_s^x) dA_s^\mu, \quad t \geq 0.$$

From the above it follows that  $M_{t \wedge \zeta}^x = M_t^x$ ,  $t \geq 0$ , and moreover, that

$$Y_t^x = Y_{T \wedge \zeta} + \int_{t \wedge \zeta}^{T \wedge \zeta} f(X_s, Y_s^x) ds + \int_{t \wedge \zeta}^{T \wedge \zeta} g(X_s, Y_s^x) dA_s^\mu - \int_{t \wedge \zeta}^{T \wedge \zeta} dM_s^x, \quad t \geq 0.$$

Letting  $T \rightarrow \infty$  and using the fact that  $Y_{T \wedge \zeta}^x \rightarrow Y_\zeta^x = 0$ ,  $P_x$ -a.s. we obtain (4.27). Thus the pair  $(Y^x, M^x)$  is a solution of (4.2).  $\square$

**Theorem 4.3.** Assume (2.9) and assume that  $f, g, \mu$  do not depend on time and satisfy (E1)–(E6).

(i) *There exists a unique solution  $v$  of problem (1.2).*

(ii) *Let*

$$M_t^x = E_x \left( \int_0^\zeta f_v(X_r) dr + \int_0^\zeta g_v(X_r) dA_r^\mu | \mathcal{F}_t \right) - v(X_0), \quad t \geq 0.$$

*Then there is a càdlàg  $(\mathcal{F}_t)$ -adapted process  $M$  such that  $M_t = M_t^x$ ,  $t \geq 0$ ,  $P_x$ -a.s. for q.e.  $x \in E$  and for q.e.  $x \in E$  the pair  $(v(X), M)$  is a unique solution of (4.2) on the space  $(\Omega, \mathcal{F}, P_x)$ . Moreover,  $v(X) \in \mathcal{D}^q(P_x)$  for  $q \in (0, 1)$  and  $M$  is a uniformly integrable martingale under  $P_x$  for q.e.  $x \in E$ .*

*Proof.* We first prove part (ii). The uniqueness of a solution of (4.2) follows easily from (E3), (E5) and the fact that  $\mu$  is positive. To see this it suffices to modify slightly the proof of [14, Proposition 3.1]. To prove the existence of a solution, we first note that by Theorem 3.2, for q.e.  $x \in E$  for every  $n \in \mathbb{N}$  there exists a unique solution  $(Y^n, M^n)$  of the BSDE (4.4) with  $\varphi \equiv 0$  on the space  $(\Omega, \mathcal{F}, P_x)$ . Since  $f(\cdot, 0) \cdot m, g(\cdot, 0) \cdot \tilde{\mu} \in \mathcal{R}(E)$ , condition (4.14) is satisfied for q.e.  $x \in E$ . Therefore, by Proposition 4.2, for q.e.  $x \in E$  there exist a solution  $(Y^x, \tilde{M}^x)$  of BSDE (4.2). In fact,  $Y^x$  is given by (4.26) and  $\tilde{M}^x$  is a càdlàg version of the martingale (4.28). Repeating step by step the proof of [14, Theorem 4.7] one can show that there is a pair of càdlàg processes  $(Y, M)$  not depending on  $x$  such that  $(Y_t, M_t) = (Y_t^x, \tilde{M}_t^x)$ ,  $t \geq 0$ ,  $P_x$ -a.s. for q.e.  $x \in E$ , and secondly, that in fact  $Y = v(X)$ , where  $v(x) = E_x Y_0$ . This shows that the pair  $(v(X), M)$  is a solution of (4.2) on the space  $(\Omega, \mathcal{F}, P_x)$  for q.e.  $x \in E$ . By Proposition 4.2,  $v(X) \in \mathcal{D}^q(P_x)$  for  $q \in (0, 1)$  and  $M$  is a uniformly integrable  $(\mathcal{F}_t)$ -martingale under  $P_x$ . This completes the proof of (ii). Part (i) follows immediately from (ii), because  $g_v \cdot \mu \in \mathcal{R}(E)$  and (4.15) is satisfied with  $Y^x$  replaced by  $v(X)$ , so for q.e.  $x \in E$  we can integrate with respect to  $P_x$  both sides of (4.2) with  $t = 0$  and  $Y^x$  replaced by  $v(X)$ .  $\square$

**Remark 4.4.** If  $g$  does not depend on the last variable  $y$  then in Theorem 4.3 we may replace the assumptions  $\tilde{\mu} \in \mathcal{R}^+(E)$ ,  $g(\cdot, 0) \cdot \tilde{\mu} \in \mathcal{R}(E)$  by the assumption  $g \cdot \tilde{\mu} \in \mathcal{R}(E)$  (see [17, Theorem 3.8]).

**Remark 4.5.** (i) By [14, Lemma 4.3] the solution  $v$  of (1.2) of Theorem 4.3 is quasi-continuous.

(ii) In addition to the hypotheses of Theorem 4.3 let us assume that  $(B, V)$  is a Dirichlet form, it is transient and  $f(\cdot, 0) \in L^1(E, m)$ ,  $g(\cdot, 0) \cdot \tilde{\mu} \in \mathcal{M}_b(E)$ , where  $\tilde{\mu}$  is determined by (2.6). Then by (4.15), Lemma 2.2, the fact that  $Y_t^x = v(X_t)$ ,  $t \geq 0$ ,  $P_x$ -a.s. and [17, Lemma 2.6] (see also [14, Lemma 5.4]),

$$\|f_v\|_{L^1(E; m)} + \|g_v \cdot \tilde{\mu}\|_{TV} \leq \|f(\cdot, 0)\|_{L^1(E; m)} + \|g(0, \cdot) \cdot \tilde{\mu}\|_{TV}.$$

Therefore, by [17, Theorem 4.2] (see also [14, Proposition 5.9]),  $f_v \in L^1(E; m)$ ,  $T_k v$  belongs to the extended Dirichlet space  $V_e$  and for every  $k > 0$ ,

$$B(T_k v, T_k v) \leq k(\|f(\cdot, 0)\|_{L^1(E; m)} + \|g(0, \cdot) \cdot \tilde{\mu}\|_{TV}).$$

Moreover, if  $(B, V)$  is a (non-symmetric) Dirichlet form satisfying the strong sector condition then by [16, Theorem 3.5]),  $v$  is a renormalized solution of problem (1.2) in the sense defined in [16]).

**Remark 4.6.** If a family  $\{B^{(t)}, t \in \mathbb{R}\}$  satisfies the assumptions of Section 2 then for every  $\lambda > 0$  the family  $\{B_\lambda^{(t)}, t \in \mathbb{R}\}$ , where  $B_\lambda^{(t)}(\varphi, \psi) = B^{(t)}(\varphi, \psi) + \lambda(\varphi, \psi)_H$ , satisfies these assumptions as well. Therefore all the results of Sections 3 and 4 apply to the operators associated with  $B_\lambda^{(t)}$  and to the Markov process associated with the form  $\mathcal{E}_\lambda$  defined by (2.4), (2.5) but with  $B^{(t)}$  replaced by  $B_\lambda^{(t)}$ .

## 5 Large time asymptotics

In this section, as in Section 4, we assume that (2.9) is satisfied and the data  $f, g, \mu$  do not depend on time. By  $L$  we denote the operator corresponding to  $(B, V)$ . We continue to write  $P_x$  for  $P_{0,x}$  and  $E_x$  for  $E_{0,x}$ , and as in Section 4, the abbreviation “q.e.” means quasi-everywhere with respect to the capacity determined by  $(B, V)$ .

Suppose that for every  $T > 0$  there exists a unique solution  $u_T$  of (3.2) with  $L$  and the data  $f, g, \mu$  satisfying the above assumptions. By Remark 3.1, putting

$$u(t, x) = \bar{u}_T(t, x) = u_T(T - t, x), \quad t \in [0, T], x \in E$$

we may define a probabilistic solution  $u$  of (1.1), i.e. solution of the problem

$$\begin{cases} \partial_t u - Lu = f(x, u) + g(x, u) \cdot \mu & \text{in } (0, \infty) \times E, \\ u(0, \cdot) = \varphi & \text{on } E. \end{cases} \quad (5.1)$$

Our goal is to prove that under suitable assumptions,  $u(t, x) \rightarrow v(x)$  as  $t \rightarrow \infty$  for q.e.  $x \in E$ , where  $v$  is a solution of (1.2), i.e. solution of the problem

$$-Lv = f(x, v) + g(x, v) \cdot \mu \quad \text{in } E. \quad (5.2)$$

We will also estimate the rate of the convergence. The proofs of these results rely on the results of Section 4. The main idea is as follows. We have

$$u(t, x) = u_T(T - t, x), \quad t \in [0, T], x \in E, \quad (5.3)$$

where  $u_T$  is a solution of the problem

$$\partial_t u_T + Lu_T = -f(x, u_T) - g(x, u_T), \quad u_T(T) = \varphi. \quad (5.4)$$

In particular, putting  $t = T$  we get  $u(T, x) = u_T(0, x)$ . Hence, by (3.5),

$$u(T, x) = E_x \left( \varphi(\mathbf{X}_{T \wedge \zeta}) + \int_0^{T \wedge \zeta} f_{u_T}(\mathbf{X}_t) dt + \int_0^{T \wedge \zeta} g_{u_T}(\mathbf{X}_t) dA_t^\mu \right), \quad (5.5)$$

because  $\zeta_\tau = T \wedge \zeta$  under the measure  $P_x$ . On the other hand,

$$v(x) = E_x \left( \int_0^\zeta f_v(\mathbf{X}_t) dt + \int_0^\zeta g_v(\mathbf{X}_t) dA_t^\mu \right). \quad (5.6)$$

Therefore our problem reduces to showing that the right-hand side of (5.5) converges to the right-hand side of (5.6) as  $T \rightarrow \infty$  and to estimating the difference between the two expressions by some function of  $T$ .



In what follows by  $(P_t)_{t \geq 0}$ ,  $(R_\alpha)_{\alpha > 0}$  denote the semigroup and the resolvent associated with the process  $\mathbb{M}^{(0)} = (X, P_x)$  with life time  $\zeta^0 = \zeta$  (see Section 2.2), i.e.

$$P_t f(x) = E_x f(X_t), \quad R_\alpha f(x) = E_x \int_0^\infty e^{-\alpha t} f(X_t) dt, \quad x \in E, f \in \mathcal{B}_b(E).$$

For  $\nu \in \mathcal{R}(E)$  we set

$$R_\alpha \nu(x) = E_x \int_0^\zeta e^{-\alpha t} dA_t^{0, \nu} = E_x \int_0^\infty e^{-\alpha t} dA_t^{0, \nu},$$

where  $A^{0, \nu}$  is the continuous AF of  $\mathbb{M}^{(0)}$  associated with  $\nu$  in the Revuz sense. Note that if  $(B, V)$  is transient then  $R_\alpha \nu$  is defined for  $\alpha = 0$ .

Before stating our main result, let us note that with the convention made at the beginning of Section 2.2,  $E_x \mathbf{1}_{\{\zeta > t\}} \psi(X_t) = P_t \psi(x)$  for  $\psi \in L^1(E; m)$ ,  $t \geq 0$ . Therefore (4.13) is equivalent to

$$\lim_{t \rightarrow \infty} P_t |\varphi|(x) = 0. \quad (5.7)$$

Clearly, assumption (4.14) is equivalent to

$$R_0 |f(\cdot, 0)|(x) + R_0 (|g(\cdot, 0)| \cdot \tilde{\mu})(x) < \infty. \quad (5.8)$$

By remarks given in Section 2.2, if  $f(\cdot, 0) \cdot m \in \mathcal{R}(E)$  and  $g(\cdot, 0) \cdot \tilde{\mu} \in \mathcal{R}(E)$  then (5.8) is satisfied for q.e.  $x \in E$ .

**Theorem 5.1.** *Assume that the assumptions of Theorem 4.3 hold, and moreover, (5.7) is satisfied. Let  $u$  be a solution of (5.1) and  $v$  be a solution of (5.2). Then*

$$\lim_{T \rightarrow \infty} u(T, x) = v(x) \quad (5.9)$$

for q.e.  $x \in E$ . In fact, for q.e.  $x \in E$ ,

$$|u(T, x) - v(x)| \leq 3P_T |\varphi|(x) + 3P_T (R_0 (|f(\cdot, 0)| + |g(\cdot, 0)| \cdot \tilde{\mu}))(x) \quad (5.10)$$

for all  $T > 0$ .

*Proof.* Let  $Y^T$  be the first component of the solution of (4.4) (with  $T = n$ ) and  $Y$  be the first component of the solution of (4.2). Since (4.14) is satisfied for q.e.  $x \in E$ , applying Proposition 4.2 we conclude that for every  $q \in (0, 1)$ ,

$$\lim_{T \rightarrow \infty} E_x |Y_0^T - Y_0|^q = 0 \quad (5.11)$$

for q.e.  $x \in E$ . On the other hand, by Theorem 3.2 and Theorem 4.3, for q.e.  $x \in E$  we have

$$Y_t^T = u_T(\mathbf{X}_t), \quad Y_t = v(\mathbf{X}_t), \quad t \geq 0, \quad P_x\text{-a.s.},$$

where  $u_T$  is a solution of (5.4) and  $v$  is a solution of (4.3). In particular, for q.e.  $x \in E$ ,

$$Y_0^T = u_T(0, x), \quad Y_0 = v(x), \quad P_x\text{-a.s.}$$

But  $u_T(0, x) = u(T, x)$  by (5.3). Hence,

$$|u(T, x) - v(x)|^q = |u_T(0, x) - v(x)|^q = E_x |Y_0^T - Y_0|^q \quad (5.12)$$

for  $T > 0$ . Therefore (5.11) implies (5.9). To show (5.10), let us first observe that by (5.11) and (5.12),

$$|u(T, x) - v(x)|^q = \lim_{m \rightarrow \infty} E_x |Y_0^T - Y_0^m|^q, \quad (5.13)$$

whereas by (4.7),

$$\begin{aligned} \lim_{m \rightarrow \infty} E_x |Y_0^T - Y_0^m|^q &\leq \frac{1}{1-q} \left( E_x \mathbf{1}_{\{\zeta > T\}} |\varphi(X_T)| + \int_{T \wedge \zeta}^{\zeta} |f(X_t, 0)| dt \right. \\ &\quad \left. + \int_{T \wedge \zeta}^{\zeta} |g(X_t, 0)| dA_t^\mu \right)^q. \end{aligned} \quad (5.14)$$

We have

$$E_x \int_{T \wedge \zeta}^{\zeta} |g(X_t, 0)| dA_t^\mu = E_x \int_T^\infty |g(\mathbf{X}_t, 0)| dA_t^\mu.$$

By the Markov property and (4.1),

$$\begin{aligned} E_x \int_T^\infty |g(\mathbf{X}_t, 0)| dA_t^\mu &= E_x E_{\mathbf{X}_T} \int_0^\infty |g(\mathbf{X}_t, 0)| dA_t^\mu \\ &= E_x E_{T, X_T} \int_0^\infty |g(X_t \circ \theta_T, 0)| dA_t^\mu, \end{aligned}$$

whereas by Lemma 2.2,

$$\begin{aligned} E_{T, X_T} \int_0^\infty |g(X_t \circ \theta_T, 0)| dA_t^\mu &= E_{T, X_T} \int_0^\infty |g(X_t \circ \theta_T, 0)| dA_t^{0, \tilde{\mu}} \circ \theta_T \\ &= E_{0, X_T} \int_0^\infty |g(X_t, 0)| dA_t^{0, \tilde{\mu}}. \end{aligned}$$

From the above it follows that

$$E_x \int_{T \wedge \zeta}^{\zeta} |g(X_t, 0)| dA_t^\mu = P_T(R_0(|g(\cdot, 0)| \cdot \tilde{\mu}))(x). \quad (5.15)$$

Similarly, since  $\int_0^t |f(X_s, 0)| ds = A_t^{f(\cdot, 0) \cdot m}$  for  $t \geq 0$ , we have

$$E_x \int_{T \wedge \zeta}^{\zeta} |f(X_t, 0)| dt = P_T(R_0(f(\cdot, 0)))(x). \quad (5.16)$$

Combining (5.13)–(5.16) yields (5.10) but with constant 3 replaced by  $(1-q)^{-1/q}$  with arbitrary  $q \in (0, 1)$ . This proves (5.10), because  $(1-q)^{-1/q} \rightarrow e$  as  $q \downarrow 0$ .  $\square$

Let  $\lambda \geq 0$  and let  $L^\lambda$  denote the operator associated with the form  $(B_\lambda, V)$ , i.e.

$$L^\lambda = L^0 - \lambda, \quad (5.17)$$

where  $L^0$  is the operator associated with  $(B_0, V) = (B, V)$ . Let  $(P_t^\lambda)$ ,  $(R_\alpha^\lambda)$  denote the semigroup and the resolvent associated with the Hunt process corresponding to  $(B_\lambda, V)$ . It is well known that for  $\psi \in L^1(E; m)$ ,  $\mu \in \mathcal{R}(E)$  we have

$$P_t^\lambda \psi(x) = e^{-\lambda t} P_t^0 \psi(x), \quad R_\alpha^\lambda \mu(x) = R_{\alpha+\lambda}^0 \mu(x)$$

for q.e.  $x \in E$ . Therefore from Theorem 5.1 we immediately get the following corollary.

**Corollary 5.2.** *Let the assumptions of Theorem 5.1 hold. Let  $u, v$  be solutions of (5.1) and (5.2), respectively, with  $L = L^\lambda$  defined by (5.17). Then for q.e.  $x \in E$ ,*

$$|u(T, x) - v(x)| \leq 3e^{-\lambda T} (P_T^0 |\varphi|(x) + P_T^0 (R_\lambda^0 (|f(\cdot, 0)| + |g(\cdot, 0)| \cdot \tilde{\mu}))(x)).$$

**Remark 5.3.** The results of Sections 3–5 can be carried over to quasi-regular forms. Indeed, if the forms  $\{B(t), t \in [0, T]\}$  are quasi-regular, then by [33, Theorem IV.2.2] there exists a special standard process  $\mathbf{M}$  properly associated in the resolvent sense with the time dependent form defined by (2.4). One can check all the results of Sections 3 and 4 hold true for such a process. This is because in their proofs the fact that  $\mathbf{M}$  is a Hunt process is not used and the results of [13] on which we rely in the proofs of Section 3 hold for quasi-regular forms  $(B^{(t)}, V)$  (see [13, Remark 4.4]). Similarly, the results of [17] on which we rely in Section 4 hold for quasi-regular form  $(B, V)$ . As a consequence, Theorem 5.1 holds true in the case of quasi regular form  $(B, V)$  (its proof for such forms requires no changes).

## 6 Applications

In this section we give four quite different examples of forms  $(B, V)$  and measures  $\mu$  for which Theorem 5.1 applies.

### 6.1 Classical local Dirichlet forms

In this subsection we assume that  $E = D$ , where  $D$  is a nonempty connected bounded open subset of  $\mathbb{R}^d$  with  $d \geq 2$ . By  $m$  we denote the Lebesgue measure on  $D$ . We consider the classical form  $(B, V)$  on  $H = L^2(D; m)$  defined as

$$B(\varphi, \psi) = \frac{1}{2} \int_D (\nabla \varphi, \nabla \psi) dx, \quad \varphi, \psi \in V. \quad (6.1)$$

We will consider two cases:  $V = H_0^1(D)$  and  $V = H^1(D)$ .

**Equations with Dirichlet boundary conditions** Let  $V = H_0^1(D)$ . It is well known that  $(B, V)$  is a regular Dirichlet form on  $H$  (see [10, Example 1.2.3]). The operator  $L$  associated with  $(B, V)$  in the sense of (3.1) is  $\frac{1}{2}\Delta$  with the Dirichlet boundary condition (see [10, Example 1.3.1]). The process  $\mathbb{M}^{(0)} = (X, P_x)$  associated with  $(B, V)$  in the resolvent sense is the Brownian motion killed upon leaving  $D$  (see [10, Example 4.4.1]). Its life time is equal to  $\tau_D = \inf\{t > 0 : X_t \notin D\}$ .

We consider the problems

$$\partial_t u - \frac{1}{2} \Delta u + h(u) |\nabla u|^2 = \mu, \quad u|_{(0, \infty) \times \partial D} = 0, \quad u(0, \cdot) = \varphi \quad (6.2)$$

and

$$-\frac{1}{2} \Delta v + h(v) |\nabla v|^2 = \mu, \quad u|_{\partial D} = 0, \quad (6.3)$$

where  $\varphi \in L^1(D; m)$ ,  $\mu \in \mathcal{M}_{0,b}^+(D)$  and  $h : \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function satisfying the “sign condition”, i.e.

$$\forall s \in \mathbb{R}, \quad h(s)s \geq 0 \quad (6.4)$$

In the above equations the gradient of the solution appears, so they are more general than the equations studied in Sections 3–5. We shall see, however, that they are closely related to equations of the forms (5.1), (5.2).

We first give definitions of probabilistic solutions of (6.2), (6.3).

**Definition.** (a) We say that  $u : (0, \infty) \times D \rightarrow \mathbb{R}$  is a probabilistic solution of (6.2) if for every  $T > 0$  the function  $\bar{u}$  defined as  $\bar{u}(t, x) = u(T - t, x)$ ,  $(t, x) \in D_T$ , is a probabilistic solution of the problem

$$\partial_t \bar{u} + \frac{1}{2} \Delta \bar{u} - h(\bar{u}) |\nabla \bar{u}|^2 = -\mu, \quad \bar{u}|_{(0, T) \times \partial D} = 0, \quad \bar{u}(T, \cdot) = \varphi \quad (6.5)$$

i.e.  $h(\bar{u}) |\nabla \bar{u}|^2 \in \mathcal{R}(D_{0, T})$  and for q.e.  $z \in D_{0, T}$ ,

$$\bar{u}(z) = E_z \left( \varphi(\mathbf{X}_{\zeta_\tau}) - \int_0^{\zeta_\tau} h(\bar{u})(\mathbf{X}_t) |\nabla \bar{u}(\mathbf{X}_t)|^2 dt + \int_0^{\zeta_\tau} dA_t^\mu \right). \quad (6.6)$$

(b) We say that  $u : D \rightarrow \mathbb{R}$  is a probabilistic solution of (6.3) if  $h(v) |\nabla v|^2 \in \mathcal{R}(E)$  and for q.e.  $x \in D$ ,

$$v(x) = E_x \left( - \int_0^\zeta h(v)(X_t) |\nabla v(X_t)|^2 dt + \int_0^\zeta dA_t^\mu \right).$$

Let

$$G(s) = 2 \int_0^s h(t) dt, \quad \Phi(s) = \int_0^s \exp(-G(t)) dt, \quad s \in \mathbb{R}$$

and let  $H : \Phi(\mathbb{R}) \rightarrow \mathbb{R}$  be defined as

$$H(s) = \exp(-G(\Phi^{-1}(s))).$$

Let us observe that (6.4) implies that  $G \geq 0$  and  $G$  is nondecreasing. Since  $\Phi$  is strictly increasing, it follows that  $H$  is nonincreasing. Of course,  $H$  is continuous and  $0 \leq H \leq 1$ , so  $H$  satisfies hypotheses (E2), (E5). Finally, let us note that  $\Phi$  is of class  $C^2$ .

In Proposition 6.1 below we show that probabilistic solutions of problems (6.2), (6.3) are closely related to the probabilistic solutions of problems

$$\partial_t w - \frac{1}{2} \Delta w = H(w) \cdot \mu, \quad w|_{(0, \infty) \times \partial D} = 0, \quad w(0, \cdot) = \Phi(\varphi) \quad (6.7)$$

and

$$-\frac{1}{2} \Delta \tilde{w} = H(\tilde{w}) \cdot \mu, \quad \tilde{w}|_{\partial D} = 0. \quad (6.8)$$

Note that assertions (ii) and (iii) of Proposition 6.1 may be viewed as probabilistic reformulation of known analytic fact relating (6.2), (6.3) to (6.7), (6.8) (see, e.g., [19]).

**Proposition 6.1.** *Assume that  $\varphi \in L^1(D; m)$ ,  $\mu \in \mathcal{M}_{0, b}^+(D)$  and  $h : D \rightarrow \mathbb{R}$  is a continuous function satisfying (6.4). Then*

- (i) *There exist unique solutions of problems (6.2) and (6.3).*
- (ii)  *$u$  is a probabilistic solution of (6.2) if and only if  $w = \Phi(u)$  is a solution of (6.7),*

(iii)  $v$  is a probabilistic solution of (6.3) if and only if  $\tilde{w} = \Phi(v)$  is a solution of (6.8).

*Proof.* We first prove (ii). Let  $w$  be a solution of (6.7) and let  $\bar{w}(t, x) = w(T - t, x)$ . By [15, Proposition 3.7], for q.e.  $z \in E_{0,T}$  the pair

$$(Y_t, Z_t) = (\bar{w}(\mathbf{X}_t), \nabla \bar{w}(\mathbf{X}_t)), \quad t \in [0, \zeta_\tau]$$

is a solution of the BSDE

$$Y_t = \Phi(\varphi(\mathbf{X}_{\zeta_\tau})) + \int_{t \wedge \zeta_\tau}^{\zeta_\tau} H(Y_s) dA_s^\mu - \int_{t \wedge \zeta_\tau}^{\zeta_\tau} Z_s dW_s, \quad t \in [0, \zeta_\tau] \quad (6.9)$$

under the measure  $P_z$ , where  $W$  is some Wiener process starting from  $z$  under  $P_z$  (In different words, in case of the form (6.1), if  $w$  is a probabilistic solution of (6.7) then the martingale  $M$  of Theorem 3.2 (with the data from (6.7)) has the representation  $M_t = \int_0^t Z_r dW_r$  with  $Z$  as above). Let  $\bar{u} = \Phi^{-1}(\bar{w})$ . Since  $\Phi^{-1}$  is of class  $C^2$ , applying Itô's formula we get

$$\begin{aligned} \bar{u}(\mathbf{X}_{\zeta_\tau}) - \bar{u}(\mathbf{X}_0) &= \Phi^{-1}(Y_{\zeta_\tau}) - \Phi^{-1}(Y_0) \\ &= \int_0^{\zeta_\tau} (\Phi^{-1})'(Y_t) dY_t + \frac{1}{2} \int_0^{\zeta_\tau} (\Phi^{-1})''(Y_t) d\langle Y \rangle_t. \end{aligned}$$

But

$$(\Phi^{-1})' = \frac{1}{\Phi'(\Phi^{-1})}, \quad (\Phi^{-1})'' = -\frac{1}{\Phi'(\Phi^{-1})^2} \cdot \Phi''(\Phi^{-1}) \cdot \frac{1}{\Phi'(\Phi^{-1})}$$

a.e. with respect to the Lebesgue measure. Hence,

$$\begin{aligned} \bar{u}(\mathbf{X}_0) &= \bar{u}(\mathbf{X}_{\zeta_\tau}) + \int_0^{\zeta_\tau} \frac{1}{\Phi'(u)} H(\bar{w}(\mathbf{X}_t)) dA_t^\mu \\ &\quad - \int_0^{\zeta_\tau} \frac{1}{\Phi'(\bar{u})} \nabla \bar{w}(\mathbf{X}_t) dB_t + \frac{1}{2} \int_0^{\zeta_\tau} \frac{\Phi''(\bar{u})}{(\Phi'(\bar{u}))^3} |\nabla \bar{w}|^2(\mathbf{X}_t) dt. \end{aligned}$$

Since  $\bar{u}(\mathbf{X}_{\zeta_\tau}) = \varphi(\mathbf{X}_{\zeta_\tau})$  and

$$\frac{\Phi''}{(\Phi')^3} = -\frac{2h}{(\Phi')^2}, \quad \Phi'(\bar{u}) = H(\bar{w}), \quad \nabla \bar{w} = \Phi'(\bar{u}) \nabla \bar{u},$$

we have

$$\bar{u}(\mathbf{X}_0) = \varphi(\mathbf{X}_{\zeta_\tau}) + \int_0^{\zeta_\tau} dA_t^\mu - \int_0^{\zeta_\tau} h(\bar{u}) |\nabla \bar{u}|^2(\mathbf{X}_t) dt - \int_0^{\zeta_\tau} \nabla \bar{u}(\mathbf{X}_t) dB_t.$$

Taking the expectation with respect to  $P_x$  we see that  $\bar{u} = \Phi^{-1}(\bar{w})$  is a probabilistic solution of (6.5). Hence  $u = \Phi^{-1}(w)$  is a probabilistic solution of (6.2).

To prove the opposite implication, let us first note that if  $u$  is a solution of (6.2) then for q.e.  $z \in E_{0,T}$  the pair

$$(\tilde{Y}_t, \tilde{Z}_t) = (\bar{u}(\mathbf{X}_t), \nabla \bar{u}(\mathbf{X}_t)), \quad t \in [0, \zeta_\tau]$$

is a solution of the BSDE

$$\tilde{Y}_t = \varphi(\mathbf{X}_{\zeta_\tau}) - \int_{t \wedge \zeta_\tau}^{\zeta_\tau} h(\tilde{Y}_s) |\tilde{Z}_s|^2 + \int_{t \wedge \zeta_\tau}^{\zeta_\tau} dA_s^\mu - \int_{t \wedge \zeta_\tau}^{\zeta_\tau} Z_s dW_s, \quad t \in [0, \zeta_\tau]$$

under the measure  $P_z$ . In case  $h(\bar{u})|\nabla\bar{u}|^2 \in L^1(D_{0,T}; m_1)$  this follows directly from [15, Proposition 3.7], while in case  $h(\bar{u})|\nabla\bar{u}|^2 \cdot m \in \mathcal{R}(D_{0,T})$  follows from [15, Proposition 3.7] by simple approximation. Put  $\bar{w} = \Phi(\bar{u})$ . Applying Itô's formula we show that the pair

$$(Y_t, Z_t) = (\bar{w}(\mathbf{X}_t), \nabla\bar{w}(\mathbf{X}_t)), \quad t \in [0, \zeta_\tau]$$

is a solution of (6.9). Therefore taking  $t = 0$  in (6.9) and then the expectation with respect to  $P_x$  shows that  $w$  is a solution of (6.5).

The proof of assertion (iii) is similar to (ii). We apply Itô's formula and the fact that in case of the form (6.1), the martingale  $M$  of Theorem 4.3 has the representation  $M_t = \int_0^t Z_s dW_s$ ,  $t \geq 0$ , with  $Z_t = \nabla v(X_t)$  if we consider equation (6.3), and with  $Z_t = \nabla\bar{w}(X_t)$  if we consider (6.8) (for the representation property for  $M$  see [12, Theorem 3.5]).

Finally, assertion (i) follows from (ii), (iii) and Theorems 3.4 and 4.3.  $\square$

**Remark 6.2.** Assume that  $\varphi \in L^1(D)$ ,  $\mu(dx) = \beta(x) dx$  for some  $\beta \in L^1(D)$  and  $h$  is a continuous function satisfying (6.4). Moreover, assume that there exist  $L, \delta > 0$  such that  $h(s)s \geq \delta$  for  $s \in \mathbb{R}$  such that  $|s| \geq L$ .

(i) In [2] it is proved that under the above assumptions there exists a weak solution  $v \in H_0^1(D)$  of (6.3) such that  $h(v)|\nabla v|^2 \in L^1(D; m)$ . A quasi-continuous version of  $v$ , which we still denote by  $v$ , is a probabilistic solution of (6.3). Indeed, since for every bounded  $w \in H_0^1$  we have  $B(v, w) = \int_D (h(v)|\nabla v|^2 + \beta)w dx$ ,  $v$  is a solution of problem (6.3) in the sense of duality (see [14, Section 5] for the definition). Therefore, by [14, Proposition 5.1],  $v$  is a probabilistic solution of (6.3).

(ii) By results proved in [31] there exists a weak solution  $\bar{u} \in L^2(0, T; H_0^1(D))$  of problem (6.5) such that  $h(\bar{u})|\nabla\bar{u}|^2 \in L^1(D_T; m_1)$ . Its quasi-continuous version is a probabilistic solution of (6.5). This follows from the fact that it is a solution of (6.5) in the sense of duality (see [13, Section 4] for the definition), and hence, by [13, Corollary 4.2], a probabilistic solution of (6.5).

**Proposition 6.3.** Let  $\varphi, h$  satisfy assumptions of Proposition 6.1 and let  $\mu(dx) = \beta(x)m(dx)$  for some nonnegative  $\beta \in L^1(D; m)$ .

(i) For q.e.  $x \in D$ ,  $u(t, x) \rightarrow v(x)$  as  $t \rightarrow \infty$ .

(ii) If, in addition,  $\varphi \geq 0$  then  $u(t, \cdot) \rightarrow v$  in  $L^1(D; m)$  as  $t \rightarrow \infty$ .

*Proof.* In the proof we adopt the notation of Proposition 6.1. Since  $H$  is bounded nonincreasing and  $\Phi$  is bounded, the initial condition  $\Phi \circ \varphi$  and coefficients  $f = 0, g = H$  satisfy the assumptions (E1)–(E6). Moreover, we shall see in the proof of Proposition 6.5 (in a more general situation where  $\Delta$  is replaced by the fractional Laplacian  $(\Delta)^{\alpha/2}$ ) that (5.7) is satisfied. Hence, by Theorem 5.1,  $w(t, x) \rightarrow \tilde{w}(x)$  as  $t \rightarrow \infty$  for q.e.  $x \in D$ . Therefore part (i) follows from Proposition 6.1 and the fact that  $\Phi^{-1}$  is continuous. To prove part (ii), let us first observe that from the formula

$$\bar{w}(z) = E_z \left( \varphi(\mathbf{X}_{\zeta_\tau}) + \int_0^{\zeta_\tau} H(\bar{w}(\mathbf{X}_t)) dA_t^\mu \right)$$

and the fact that  $H \geq 0$  it follows immediately that if  $\varphi \geq 0$  then  $\bar{w} \geq 0$ . Since  $\Phi(s) \geq 0$  for  $s \geq 0$ , we conclude from this and Proposition 6.1 that  $\bar{u} \geq 0$ . Consequently,  $h(\bar{u}) \geq 0$

since  $h$  satisfies (6.4). Therefore from (6.6) it follows that for q.e.  $z \in E_{0,T}$ ,

$$\bar{u}(z) \leq E_z \left( \varphi(\mathbf{X}_{\zeta_\tau}) + \int_0^{\zeta_\tau} dA_t^\mu \right) := \bar{\hat{u}}(s, x).$$

The function  $\hat{u}$  defined as  $\hat{u}(t, x) = \bar{\hat{u}}(T - t, x)$ ,  $(t, x) \in D_T$ , is a solution of (6.2) with  $h \equiv 0$ . By Theorem 5.1,  $\hat{u}(t, x) \rightarrow \hat{v}(x)$  as  $t \rightarrow \infty$  for q.e.  $x \in D$ , where  $\hat{v}$  is a solution of (6.3) with  $h \equiv 0$ . In fact, by (5.10) (see the proof of Proposition 6.5 for details),

$$|\hat{u}(t, x) - \hat{v}(x)| \leq Ct^{-d/2}(\|\varphi\|_{L^1(D; m)} + \|\beta\|_{L^1(D; m)})$$

for q.e.  $x \in D$ . Hence  $\hat{u}(t, \cdot) \rightarrow \hat{v}$  in  $L^1(D; m)$  as  $t \rightarrow \infty$ . From this and the fact that  $0 \leq u(t, \cdot) \leq \hat{u}(t, \cdot)$  it follows that the family  $\{u(t, \cdot)\}$  is uniformly integrable, which together with (i) proves (ii).  $\square$

By using a completely different method part (ii) of the above proposition was proved in [19, Theorem 3.3] under the assumption that  $h \in C^1(\mathbb{R})$  and  $h'(s) > 0$  for  $s \in \mathbb{R}$ .

**Equations with Neumann boundary conditions** Let  $D$  be a bounded Lipschitz domain of  $\mathbb{R}^d$ ,  $d \geq 3$ . Set  $E = \bar{D}$ . Let  $H = L^2(\bar{D}; m)$ , where  $m$  is the Lebesgue measure on  $\bar{D}$ , and let  $V = H^1(D)$ . It is known that  $(B, V)$  defined by (6.1) is a regular Dirichlet form on  $H$  (see [10, Example 4.5.3]). The operator  $L$  associated with  $(B, V)$  in the sense of (3.1) is  $\frac{1}{2}\Delta$  with the Neumann boundary condition, while the process  $\mathbb{M}^{(0)} = (X, P_x)$  (with life time  $\zeta = \infty$ ) associated with  $(B, V)$  in the resolvent sense is the reflecting Brownian motion on  $\bar{D}$  (see [10, Example 4.5.3]).

Let  $\nu$  denote the surface measure on  $\partial D$ . Then for  $\nu$ -a.e.  $x \in \partial D$  there exists a unit inward normal vector  $\mathbf{n}(x) = (\mathbf{n}_1(x), \dots, \mathbf{n}_d(x))$  (see [10, Example 5.2.2]). We consider the Neumann problems

$$\partial_s u - \frac{1}{2}\Delta u + \lambda u = f(\cdot, u), \quad \frac{\partial u}{\partial \mathbf{n}}|_{(0, \infty) \times \partial D} = g(x, u), \quad u(0, \cdot) = \varphi \quad (6.10)$$

and

$$-\frac{1}{2}\Delta v + \lambda v = f(\cdot, v), \quad \frac{\partial v}{\partial \mathbf{n}}|_{\partial D} = g(\cdot, v), \quad (6.11)$$

where  $\frac{\partial u}{\partial \mathbf{n}} = \sum_{i=1}^d \mathbf{n}_i \frac{\partial u}{\partial x_i}$ . It is known (see [10, Example 5.2.2]) that for every  $x \in \bar{D}$  the process  $X$  has under  $P_x$  the representation

$$X_t^i = X_0^i + B_t^i + \frac{1}{2} \int_0^t \mathbf{n}_i(X_s) dl_s, \quad t \geq 0, \quad P_x\text{-a.s.}, \quad (6.12)$$

where  $B = (B^1, \dots, B^d)$  is a  $d$ -dimensional standard Brownian motion and  $l$  is the local time of  $X$  on the boundary  $\partial D$ . In fact,  $\nu \in S_{00}(\bar{D})$  and

$$A^\nu = l. \quad (6.13)$$

By (6.12) and (6.13) the probabilistic solution of (6.10) (see [25, Section 4]) coincides with the probabilistic solution of

$$\partial_t u - Lu + \lambda u = f(x, u) + g(x, u) \cdot \nu, \quad u(0, \cdot) = \varphi \quad (6.14)$$

and the probabilistic solution of (6.11) (see [25, Section 5]) coincides with the probabilistic solution of

$$-Lw + \lambda w = f(x, w) + g(x, w) \cdot \nu. \quad (6.15)$$

**Proposition 6.4.** *Let  $\varphi, f, g$  satisfy the assumptions of Theorem 4.3, and moreover,  $f(\cdot, 0) \in L^1(D; m)$ ,  $g(\cdot, 0) \in L^\infty(D; m)$ . Let  $u$  be a solution of (6.14) and  $v$  be a solution of (6.15). Then for every  $\lambda > 0$  there is  $C > 0$  depending only on  $d$  such that for q.e.  $x \in \bar{D}$ ,*

$$|u(t, x) - v(x)| \leq C e^{-\lambda t} t^{-d/2} \left( \|\varphi\|_{L^1(D; m)} + \lambda^{-1} \|f(\cdot, 0)\|_{L^1(D; m)} + (1 \vee \lambda^{-1}) m(D) \|g(\cdot, 0)\|_\infty \|R_1^0 \nu\|_\infty \right), \quad t > 0.$$

*Proof.* By [1, Theorem 3.1] (see also [1, Lemma 4.3]) there is  $C > 0$  depending only on  $d$  such that for every  $\psi \in L^1(D; m)$ ,

$$\sup_{x \in \bar{D}} P_t^0 |\psi|(x) \leq C t^{-d/2} \|\psi\|_{L^1(D; m)}, \quad t > 0.$$

Moreover,

$$\|R_\lambda^0 f(\cdot, 0)\|_{L^1(D; m)} = (f(\cdot, 0), R_\lambda^0 1) = \lambda^{-1} \|f(\cdot, 0)\|_{L^1(D; m)}.$$

Since  $\nu \in S_{00}$ ,  $\|R_1^0 \nu\|_\infty < \infty$ . By the resolvent equation (see [10, Lemma 5.1.5]),

$$R_\lambda^0 \nu = R_1^0 \nu + (1 - \lambda) R_\lambda^0 (R_1^0 \nu).$$

Hence  $\|R_\lambda^0 (g(\cdot, 0) \cdot \nu)\|_{L^1(D; m)} \leq \|g(\cdot, 0)\|_\infty \|R_1^0 \nu\|_{L^1(D; m)} \leq m(D) \|g(\cdot, 0)\|_\infty \|R_1^0 \nu\|_\infty$  if  $\lambda \geq 1$  and

$$\begin{aligned} \|R_\lambda^0 (g(\cdot, 0) \nu)\|_{L^1(D; m)} &\leq m(D) \|g(\cdot, 0)\|_\infty (\|R_1^0 \nu\|_\infty + (1 - \lambda) \lambda^{-1} \|R_1^0 \nu\|_\infty) \\ &= \lambda^{-1} m(D) \|g(\cdot, 0)\|_\infty \|R_1^0 \nu\|_\infty \end{aligned}$$

if  $\lambda < 1$ . The proposition follows immediately from the above estimates and Corollary 5.2.  $\square$

## 6.2 Nonlocal Dirichlet forms

Let  $E = \mathbb{R}^d$  with  $d \geq 2$ ,  $m$  be the Lebesgue measure on  $E$  and  $\alpha \in (0, 2)$ . Let us consider the form

$$B(u, v) = \int_{\mathbb{R}^d} \hat{u}(x) \hat{v}(x) |x|^\alpha dx, \quad u, v \in V, \quad (6.16)$$

where  $\hat{u}$  denotes the Fourier transform of  $u$  and

$$V = \{u \in L^2(\mathbb{R}^d; m); \int_{\mathbb{R}^d} |\hat{u}(x)|^2 |x|^\alpha dx < \infty\}.$$

It is known that  $(B, V)$  is a regular Dirichlet form on  $L^2(\mathbb{R}^d; m)$  (see [10, Example 1.4.1]). The operator  $L$  associated with  $(B, V)$  is the fractional Laplacian  $\Delta^{\alpha/2}$  and the Markov process  $\mathbb{M}^{(0)} = (X, P_x)$  (with life time  $\zeta = \infty$ ) associated with  $(B, V)$  is a symmetric stable process of index  $\alpha$ .

Let  $D \subset \mathbb{R}^d$ ,  $d \geq 2$ , be a nonempty open bounded connected set. Set  $L_D^2(\mathbb{R}^d; m) = \{u \in L^2(\mathbb{R}^d; m) : u = 0 \text{ a.e. on } D^c\}$ ,  $V_D = \{u \in D(B) : \tilde{u} = 0 \text{ q.e. on } D^c\}$ , where  $\tilde{u}$  is a quasi-continuous version of  $u$ . By [10, Theorem 4.4.3], the form  $(B, V_D)$  is a regular Dirichlet form on  $L_D^2(\mathbb{R}^d; m)$ , and by [10, Theorem 4.4.4], if  $(B, V)$  is transient then  $(B, V_D)$  is transient, too.

In what follows by  $\mathbb{M}_D^{(0)}$  we denote the part of the process  $\mathbb{M}^{(0)}$  on  $D$  (see [10, Section 4.4]), by  $\zeta_D$  its life time and by  $(P_t^D)$  the semigroup associated with  $\mathbb{M}_D^{(0)}$ .



**Proposition 6.5.** *Let  $\varphi, f, g$  satisfy the assumptions of Theorem 4.3, and moreover,  $f(\cdot, 0) \in L^1(D; m)$ ,  $g(\cdot, 0) \cdot \tilde{\mu} \in \mathcal{M}_{0,b}(D)$ . Let  $u$  be a solution of (5.1) and  $v$  be a solution of (5.2). Then there exists  $C > 0$  depending only on  $d, \alpha$  such that q.e.  $x \in D$ ,*

$$|u(t, x) - v(x)| \leq Ct^{-d/\alpha} (\|\varphi\|_{L^1(D; m)} + (m(D))^{\alpha/d} \|f(\cdot, 0)\|_{L^1(D; m)} + (m(D))^{\alpha/d} (|g(\cdot, 0)| \cdot \tilde{\mu})(D)), \quad t > 0. \quad (6.17)$$

*Proof.* Let  $\mathbb{M}_D^{(0)}$  we denote the part of the process  $\mathbb{M}^{(0)}$  on  $D$  (see [10, Section 4.4]),  $\zeta_D$  denote the life time of  $\mathbb{M}_D^{(0)}$  and let  $(P_t^0)$ ,  $(R_\alpha^0)$  denote the semigroup and the resolvent associated with  $\mathbb{M}_D^{(0)}$ . By  $p$  let us denote the transition density of the process  $\mathbb{M}^{(0)}$ . From the fact that  $p(t, x, y) = p(t, 0, x - y)$  and the scaling property  $p(t, 0, x) = t^{-d/\alpha} p(1, 0, t^{-1/\alpha} x)$  it follows that

$$p(t, x, y) \leq Ct^{-d/\alpha}, \quad t > 0 \quad (6.18)$$

with  $C = \sup_{x \in \mathbb{R}^d} p(1, 0, x)$ . Hence,

$$P_t^0 \varphi(x) \leq Ct^{-d/\alpha} \|\varphi\|_{L^1(D; m)}, \quad t > 0. \quad (6.19)$$

By (6.18) and [6, Theorem 1] (see also the proof of [7, Theorem 1.17]),

$$\sup_{x \in D} E_x \zeta_D^0 \leq c(m(D))^{\alpha/d}$$

for some  $c > 0$  depending only on  $\alpha, d$ . By (6.19),

$$P_t^0 (R_0^0 (|g(\cdot, 0)| \cdot \tilde{\mu}))(x) \leq Ct^{-d/\alpha} \|R_0^0 (|g(\cdot, 0)| \cdot \tilde{\mu})\|_{L^1(D; m)}.$$

Since

$$\begin{aligned} \|R_0^0 (|g(\cdot, 0)| \cdot \tilde{\mu})\|_{L^1(D; m)} &= \int_D R_0^0 1(x) |g(x, 0)| \tilde{\mu}(dx) \\ &= \int_D E_x \zeta_D^0 |g(x, 0)| \tilde{\mu}(dx) \leq c(m(D))^{\alpha/d} (|g(\cdot, 0)| \cdot \tilde{\mu})(D), \end{aligned}$$

we have

$$P_t^0 (R_0^0 (|g(\cdot, 0)| \cdot \tilde{\mu}))(x) \leq c(\alpha, d) (m(D))^{\alpha/d} t^{-d/\alpha} (|g(\cdot, 0)| \cdot \tilde{\mu})(D). \quad (6.20)$$

Putting  $g = 1$  and  $\mu = f(\cdot, 0) \cdot m$  in the above estimate we get

$$P_t^0 (R_0^0 |f(\cdot, 0)|)(x) \leq c(\alpha, d) (m(D))^{\alpha/d} t^{-d/\alpha} \|f(\cdot, 0)\|_{L^1(D; m)}. \quad (6.21)$$

Substituting (6.19)–(6.21) into (5.10) we get the desired estimate.  $\square$

Let us assume additionally that  $D$  has a  $C^{1,1}$  boundary and  $d \geq 3$ . Then by [18, Proposition 4.9] there exist constants  $0 < c_1 < c_2$  depending only on  $d, \alpha, D$  such that

$$c_1 \delta^{\alpha/2}(x) \leq R_0^0 1(x) \leq c_2 \delta^{\alpha/2}(x), \quad x \in D,$$

where  $\delta(x) = \text{dist}(x, \partial D)$ . It follows that if

$$\int_D \delta^{\alpha/2}(x) |g(\cdot, 0)| \tilde{\mu}(dx) := K < \infty, \quad (6.22)$$

then (6.21) holds with  $|\mu(D)|$  replaced by  $K$ . Therefore under the above assumptions on  $D$  the proof of Proposition 6.5 shows the following proposition.

**Proposition 6.6.** *Let the assumptions of Proposition 4.1 hold, and moreover  $f \in L^1(D; m)$ ,  $|g(\cdot, 0)| \cdot \tilde{\mu}$  satisfies (6.22). Then (6.17) holds true with  $(m(D))^{\alpha/d}(|g(\cdot, 0)| \cdot \mu)(D)$  replaced by  $K$ .*

**Remark 6.7.** (i) An analogue of Proposition 6.5 holds true for  $D$  as before and form (6.16) replaced by any regular transient symmetric Dirichlet form  $(B, V)$  on  $L^2(\mathbb{R}^d; dx)$  whose semigroup possesses a kernel  $p$  satisfying uniform estimate of the form (6.18) with  $\alpha/d$  replaced by  $\kappa$ , i.e.

$$p(t, x, y) \leq Ct^{-\kappa}, \quad t > 0 \quad (6.23)$$

for some  $C, \kappa > 0$ . Indeed, an inspection of the proof of Proposition 6.5 shows that for such a form estimate (6.17) holds with  $\alpha/d$  replaced by  $\kappa$ . A characterization of symmetric Dirichlet forms satisfying (6.23) in terms of Dirichlet form inequalities of Nash's type is given in [4]. For a concrete example of a class of forms satisfying (6.23) and containing the form (6.16) as a special case see [4, Remark 2.15]. For similar examples see [5, Examples 6.7.14, 6.7.16].

### 6.3 Local semi-Dirichlet forms

Let  $D \subset \mathbb{R}^d$ ,  $m, H$  be as in Section 6.1 and let  $a : D \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d$ ,  $b : D \rightarrow \mathbb{R}^d$  be measurable functions such that for every  $x \in D$ ,

$$\lambda^{-1}|\xi|^2 \leq \sum_{i,j=1}^d a_{ij}(x)\xi_i\xi_j \leq \lambda|\xi|^2, \quad a_{ij}(x) = a_{ji}(x), \quad \sum_{i=1}^d |b_i(x)|^2 \leq \lambda$$

for some  $\lambda \geq 1$ . Set  $V = H_0^1(D)$  and

$$B(\varphi, \psi) = \sum_{i,j=1}^d \int_D a_{ij}(x) \frac{\partial \varphi}{\partial x_i} \frac{\partial \psi}{\partial x_j} dx + \sum_{i=1}^d \int_D b_i(x) \frac{\partial \varphi}{\partial x_i} \psi(x) dx, \quad \varphi, \psi \in V.$$

Of course, the operator  $L$  determined by  $(B, V)$  has the form

$$L = \sum_{i,j=1}^d \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial}{\partial x_j}) + \sum_{i=1}^d b_i(x) \frac{\partial}{\partial x_i}. \quad (6.24)$$

By [24, Theorem 1.5.3],  $(B, V)$  is a regular lower bounded semi-Dirichlet form on  $H$ . It is well known that the transition density  $p$  of the process associated with  $(B, V)$  has the property that  $p(t, x, y) \leq Ct^{-d/2}$ ,  $t > 0$ , for some  $C > 0$  (i.e. (6.18) with  $\alpha = 2$  is satisfied). Therefore there is an analogue of Proposition 6.5 for equations involving the operator  $L$  defined by (6.24).

### Acknowledgements

Research supported by National Science Centre Grant No. 2012/07/B/ST1/03508.

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